Improving Volatility Forecasts Using Market-elicited Ambiguity Aversion Information

Raymond H.Y. So† and Tarik Driouchi‡

TECHNICAL APPENDIX – Option Pricing under Ambiguity (Based on Driouchi, Trigeorgis and So, 2016)

Let B be the price of a riskless bond with instantaneous rate of return r such that:

$$\frac{dB}{B} = rdt \qquad (A1)$$

Let O be the price of a contingent-claim (e.g., a European call or put option on the S&P index) which depends only on S and time t, $O(S,t)$. From Ito's lemma and Eq. (1), the dynamics of option price O can be written ($\forall\, m\ \in]-1,1[\ ,\forall\, s\ \in]0,1]$) as:

$$dO(S,t) = \frac{\partial O}{\partial t}dt + \frac{\partial O}{\partial S}[(\mu + m\sigma)Sdt + (s\sigma)SdZ] + \frac{1}{2}\frac{d^2O}{dS^2}([(s\sigma)SdZ]^2 + [(\mu + m\sigma)Sdt]^2 + [(s\sigma)SdZ \times (\mu + m\sigma)Sdt]) \qquad (A2)$$

This simplifies to:

$$dO(S,t) = \left[\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S}(\mu + m\sigma)S + S^2\frac{1}{2}\frac{d^2O}{dS^2}(s\sigma)^2\right]dt + \frac{\partial O}{\partial S}(s\sigma)SdZ \qquad (A3)$$

Using Eq. (1), the level(s) of marginal utility in the economy ξ under Choquet ambiguity is:

---

† King's Business School, King's College London, University of London. Bush House, 30 Aldwych, London WC2B 4BG, United Kingdom; Tel: +44 020 78484943; Email: raymond.so@kcl.ac.uk
‡ King's Business School, King's College London, University of London. Bush House, 30 Aldwych, London WC2B 4BG, United Kingdom.

$$\frac{d\xi}{\xi} = [mg(\xi,S) + f(\xi,S)]dt + sg(\xi,S)dz \qquad (A4)$$

This results from standard economic dynamics $\frac{d\xi}{\xi} = f(\xi,S)dt + g(\xi,S)dW$ (see Harrison and Kreps, 1979) and the characteristics of *W* in the Choquet ambiguity universe. Functions *g* and *f* help derive the pricing kernel under uncertainty. Thus:

$$d(\xi B) = \xi(rBdt) + B\big[\big(mg(\xi,S) + f(\xi,S)\big)\xi dt + sg(\xi,S)\xi dz\big]$$
$$= \xi B\big[\big(r + mg(\xi,S) + f(\xi,S)\big)dt + sg(\xi,S)\xi dz\big] \qquad (A5)$$

Applying martingale theory, the drift (dt) term is set to zero. This implies:

$$r + mg(\xi,S) + f(\xi,S) = 0 \text{ or } f(\xi,S) = -r - mg(\xi,S) \qquad (A6)$$

Following a similar procedure for *S*:

$$d(\xi S) = \xi dS + Sd\xi + d < \xi,S > \qquad (A7)$$

$$d(\xi S) = \xi S[(\mu + m\sigma)dt + s\sigma dz] + S\xi\{[mg(\xi,S) - r - mg(\xi,S)]dt + sg(\xi,S)dz\} + s^2\sigma\xi S g(\xi,S)dt$$

$$= \xi S[(\mu + m\sigma) - r + s^2\sigma g(\xi,S)]dt + S\xi[s\sigma + sg(\xi,S)]dz \qquad (A8)$$

Setting the drift term to zero, we obtain the ambiguity-adjusted Sharpe ratio $g(\xi,S)$:

$$(\mu + m\sigma) - r + s^2\sigma g(\xi,S) = 0 \text{ or } g(\xi,S) = \frac{[r - (\mu + m\sigma)]}{s^2\sigma} \qquad (A9)$$

The market pricing kernel follows Harrison and Kreps (1979) dynamics but, due to market incompleteness, multiple marginal utility levels and Knightian uncertainty, *f* and *g* are not unique as they are affected by investors' ambiguity parameters *m* and *s*. This means that Choquet ambiguity impacts the fundamental component of the market pricing kernel (via parameters *m* and *s*) but not the purely sentimental element (see Cochrane, 2001; Shefrin, 2005). Relaxing this general (market incompleteness) assumption reduces to the perfect replication or risk-neutral case of Black-Scholes (1973) OPM. Using the results from Eqs. (A6) and (A9):

$$\frac{d\xi}{\xi} = f(\xi, S)dt + g(\xi, S)dz$$

$$= -r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right\}dt + \left(\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right)dz \qquad (A10)$$

Consider the value of a call or put option O written on underlying stock index S (with dividend yield δ).

$$d(\xi O) = \xi dO + Od\xi + d < \xi, O >$$

$$= \xi\left\{\left[\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S}(\mu - \delta + m\sigma)S + S^2\frac{1}{2}\frac{d^2O}{dS^2}(s\sigma)^2\right]dt + \frac{\partial O}{\partial S}(s\sigma)Sdz\right\}$$

$$+ \xi O\left[-r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right\}dt + \left(\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right)dz\right]$$

$$+ \xi\left[\left(\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right)\frac{\partial O}{\partial S}(s\sigma)Sdt\right] \qquad (A11)$$

Setting the drift (dt) term of the option to zero results in the fundamental equation for pricing derivatives or contingent-claims $O_{c/p}$:

$$\xi\left[\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S}(\mu - \delta + m\sigma)S + S^2\frac{1}{2}\frac{d^2O}{dS^2}(s\sigma)^2\right]dt + \xi O\left[-r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right\}dt\right]$$
$$+ \xi\left[\left(\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right)\frac{\partial O}{\partial S}(s\sigma)Sdt\right] = 0 \qquad (A12)$$

Solving Eq. (A12) for European options written on S leads to Eqs. (3-5).

References for the Technical Appendix:

Cochrane, J.H. 2001. *Asset Pricing*. Princeton, NJ: Princeton University Press.

Harrison, J.M., and D.M. Kreps. 1979. Martingale and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20, 381-408.

Shefrin, H. 2005. *A Behavioral Approach to Asset Pricing*. NYC: Elsevier Academic Press.