Improving Volatility Forecasts Using Market-elicited Ambiguity Aversion Information
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TECHNICAL APPENDIX – Option Pricing under Ambiguity (Based on Driouchi, Trigeorgis and So, 2016)

Let $B$ be the price of a riskless bond with instantaneous rate of return $r$ such that:

$$\frac{dB}{B} = r\,dt \quad (A1)$$

Let $O$ be the price of a contingent-claim (e.g., a European call or put option on the S&P index) which depends only on $S$ and time $t$, $O(S,t)$. From Ito’s lemma and Eq. (1), the dynamics of option price $O$ can be written ($\forall m \in [-1,1], \forall s \in [0,1]$) as:

$$dO(S,t) = \frac{\partial O}{\partial t} \, dt + \frac{\partial O}{\partial S}[(\mu + m\sigma)S\,dt + (s\sigma)S\,dZ] + \frac{1}{2} \frac{d^2 O}{dS^2}[(s\sigma)S\,dZ]^2$$

$$+ [(\mu + m\sigma)S\,dt]^2 + [(s\sigma)S\,dZ \times (\mu + m\sigma)S\,dt]] \quad (A2)$$

This simplifies to:

$$dO(S,t) = \left[\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu + m\sigma)S + \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right] dt + \frac{\partial O}{\partial S} (s\sigma)S\,dZ \quad (A3)$$

Using Eq. (1), the level(s) of marginal utility in the economy $\xi$ under Choquet ambiguity is:

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\[ \frac{d\xi}{\xi} = [mg(\xi, S) + f(\xi, S)]dt + sg(\xi, S)dz \]  

(A4)

This results from standard economic dynamics \( \frac{df}{\xi} = f(\xi, S)dt + g(\xi, S)dW \) (see Harrison and Kreps, 1979) and the characteristics of \( W \) in the Choquet ambiguity universe. Functions \( g \) and \( f \) help derive the pricing kernel under uncertainty. Thus:

\[ d(\xi B) = \xi (rBdt) + B[(mg(\xi, S) + f(\xi, S))\xi dt + sg(\xi, S)\xi dz] \]

\[ = \xi B[(r + mg(\xi, S) + f(\xi, S))dt + sg(\xi, S)\xi dz] \]  

(A5)

Applying martingale theory, the drift \((dt)\) term is set to zero. This implies:

\[ r + mg(\xi, S) + f(\xi, S) = 0 \text{ or } f(\xi, S) = -r - mg(\xi, S) \]  

(A6)

Following a similar procedure for \( S \):

\[ d(\xi S) = \xi dS + Sd\xi + d < \xi, S > \]  

(A7)

\[ d(\xi S) = \xi S[\mu + m\sigma]dt + s\sigma dz + S\xi[\mu \xi Sg(\xi, S)]dt + sg(\xi, S)\xi dz + s^2\sigma^2\xi Sg(\xi, S)dt \]

\[ = \xi S[\mu + m\sigma - r + s^2\sigma g(\xi, S)]dt + S\xi[s\sigma + g(\xi, S)]dz \]  

(A8)

Setting the drift term to zero, we obtain the ambiguity-adjusted Sharpe ratio \( g(\xi, S) \):

\[ (\mu + m\sigma) - r + s^2\sigma g(\xi, S) = 0 \text{ or } g(\xi, S) = \frac{[r - (\mu + m\sigma)]}{s^2\sigma} \]  

(A9)

The market pricing kernel follows Harrison and Kreps (1979) dynamics but, due to market incompleteness, multiple marginal utility levels and Knightian uncertainty, \( f \) and \( g \) are not unique as they are affected by investors’ ambiguity parameters \( m \) and \( s \). This means that Choquet ambiguity impacts the fundamental component of the market pricing kernel (via parameters \( m \) and \( s \)) but not the purely sentimental element (see Cochrane, 2001; Shefrin, 2005). Relaxing this general (market incompleteness) assumption reduces to
the perfect replication or risk-neutral case of Black-Scholes (1973) OPM. Using the results from Eqs. (A6) and (A9):

\[
\frac{d\xi}{\xi} = f(\xi, S) dt + g(\xi, S) dz
\]

\[
= -r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right\} dt + \left\{\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right\} dz
\]  

(A10)

Consider the value of a call or put option \(O\) written on underlying stock index \(S\) (with dividend yield \(\delta\)).

\[
d(\xi O) = \xi dO + O d\xi + d < \xi, O >
\]

\[
= \xi \left\{\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu - \delta + m\sigma)S + S^2 \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right\} dt + \frac{\partial O}{\partial S} (s\sigma)S dz
\]

\[
+ \xi O \left\{-r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right\} dt + \left\{\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right\} dz\right\}
\]

\[
+ \xi \left\{\left(\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right) \frac{\partial O}{\partial S} (s\sigma)S dt\right\}
\]

(A11)

Setting the drift (dt) term of the option to zero results in the fundamental equation for pricing derivatives or contingent-claims \(O_{\text{e-p}}\):

\[
\xi \left\{\frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu - \delta + m\sigma)S + S^2 \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right\} dt + \xi O \left\{-r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right\} dt\right\}
\]

\[
+ \xi \left\{\left(\frac{[r - (\mu + m\sigma)]}{s^2 \sigma}\right) \frac{\partial O}{\partial S} (s\sigma)S dt\right\} = 0
\]  

(A12)

Solving Eq. (A12) for European options written on \(S\) leads to Eqs. (3-5).

References for the Technical Appendix:
