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Endogenous Financial Constraint and Investment-Cash-Flow Sensitivity

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Abstract

This paper studies a dynamic investment model with moral hazard. The moral hazard problem implies an endogenous financial constraint on investment that makes the firm's investment sensitive to cash flows. I show that the production technology and the severity of the moral hazard problem substantially affect the dependence of the investment-cash-flow sensitivity on the financial constraint. Specifically, if the production technology exhibits almost constant returns to scale in capital or the moral hazard problem is relatively severe, the dependence is *negative*. Otherwise, the pattern is reversed to some extent. Moreover, the calibrated benchmark model can quantitatively account for the negative dependence of investment and Tobin's Q on size and age observed in the data.

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1 Introduction

The literature documents a significant correlation between firms' investments and their cash flow after controlling for Tobin's Q, which contradicts the traditional Q theory in a frictionless economy. People attribute this investment-cash-flow sensitivity to the firm's financial constraint on investment implied by various frictions. The subject of research over the past several decades studies the dependency of investment cash-flow sensitivity on the tightness of the financial constraint. Understanding this dependence enables us to see whether this sensitivity is a quantitative measure of the financial constraint, which substantially affects firm behavior but is difficult to observe. However, researchers obtains mixed results about this dependence. For example, Fazzari, Hubbard, and Petersen (1988) and Gilchrist and Himmelberg (1995) find that more financially constrained firms exhibit greater investment-cash-flow sensitivities,[1] whereas Kaplan and Zingales (1997) and Cleary (1999) find the opposite pattern. Until now, this research question has remained open.[2] Surprisingly, no attempt has been made to understand whether a unique correlation exists between investment-cash-flow sensitivity and the financial constraint in general or to determine what factors make this correlation positive or negative. Some of these important factors could potentially vary across industries, countries, or time.

In this paper, I introduce moral hazard into an otherwise standard firm investment model where investment is sensitive to cash flows because of an endogenous financial constraint on investment. Under this theoretical framework, I quantitatively show that whether the magnitude of the investment-cash-flow sensitivity increases or decreases with the tightness of the financial constraint depends on two important factors: (1) the returns to scale in capital of the production technology and (2) the severity of the moral hazard problem. Specifically, if the returns to scale in capital are close to constant or the moral hazard problem is relatively severe, the sensitivity *decreases* with the financial constraint; otherwise, the dependence is reversed to some extent.

In my model, the economy consists of a large number of entrepreneurs, each of whom is endowed with a technology that allows him to produce consumption goods from capital over a long time horizon. The cash flows generated by the firms are subject to random shocks, and the firms incur temporary losses. Since the entrepreneurs do not have initial wealth, they ask an investor for financing to cover their losses so that their firms can maintain capital input. Because the cash flow shocks are not observable to the investors, so that the entrepreneurs could misreport the temporary losses and divert cash flows, creating moral

[1]Based on this result, some researchers use investment-cash-flow sensitivity as an indicator of the financial constraint (e.g. Hoshi, Kashyap, and Scharfstein (1991) and Almeida and Campello (2007)).

[2]Kadapakkam, Kumar, and Riddick (1998) and Vogt (1994) find that large firms that seem to be less financially constrained exhibit greater investment-cash-flow sensitivity.

hazard. Therefore, to deter hidden diversions, an entrepreneur's stake in the firm, the fraction of the firm's future cash flows that belongs to him, is sensitive to the reported cash flows. The entrepreneur is protected by limited liability so that the firm has to be liquidated when this stake reaches zero after a sequence of negative cash flow shocks. Since liquidation is inefficient, an agency cost of incentive provisions arises, which implies a financial constraint on capital input. This is because a higher level of capital input increases the cash flows overseen by the entrepreneur and thus requires more intense incentive provisions, which raise the liquidation probability. As a result of the pay-performance sensitivity, positive cash flow shocks raise the entrepreneur's stake in the firm, relax the financial constraint, and allow the firm to input a higher and more efficient level of capital. However, negative shocks lower this stake, tighten the financial constraint, and force the firm to cut capital input. Consequently, the model endogenously generates an investment-cash-flow sensitivity.

If the production technology exhibits almost constant returns to scale, the marginal product of capital does not diminish at high capital levels. Therefore, when the financial constraint is relaxed upon a positive cash-flow shock, capital input increases significantly. In addition, a high level of capital requires more intense incentive provisions so that the financial constraint is more responsive to cash flow shocks. Consequently, less constrained firms, which deploy more capital, exhibit larger investment-cash-flow sensitivities. Clearly, if the marginal product of capital diminishes significantly at high levels of capital, in a less constrained firm, the investment would respond less to cash flow shocks. I measure the severity of the moral hazard problem by the volatility of the unobservable cash flow shocks, the noise in the entrepreneur's cash flow reports. If the noise level is high, a less constrained firm is still subject to a significant liquidation probability so that cash flow shocks still have significant impacts on the financial constraint. Given the high levels of capital input and the incentive provisions, investment is more responsive to cash flow shocks in this firm. However, if the noise level is low, in a less constrained firm, cash flow shocks have a weaker influence on the financial constraint and on investment. My results suggest that it is important to control for the production technology and the severity of the moral hazard problem when studying how investment-cash-flow sensitivity depends on the financial constraint.

Under my framework, a young firm is subject to a tighter financial constraint. In the early stage of its life cycle, the firm's growth and investment are driven by the relaxation of the financial constraint and the progress of productivity. In the late stage, when the firm grows larger and older, the agency cost vanishes and the financial constraint is relaxed so that its investment and growth rely only on the progress of productivity. Therefore, large and old firms invest less, compared with their small and young counterparts. Since Tobin's Q is approximately the marginal product of capital, it decreases as the firm grows large. The negative dependence of investment and Tobin's Q on size and age are consistent with the data.

The main topic of this paper relates to whether investment-cash-flow sensitivity is a quantitative measure of the financial constraint. The theoretical framework builds on the rapidly growing literature on continuous-time dynamic contract design models (DeMarzo and Sannikov (2006), Biais, Mariotti, Rochet, and Villeneuve (2010), Williams (2011), DeMarzo, Fishman, He, and Wang (2012), and Zhu (2013)). In contrast to these studies, this paper studies the implications of the moral hazard problem on the pattern of investment-cash-flow sensitivity and the endogenous financial constraint on investment. This paper also relates to studies of the dependence of investment-cash-flow sensitivity on firm characteristics including Gomes (2001), Alti (2003), Moyen (2004), Lorenzoni and Walentin (2007), and Abel and Eberly (2011). None of these papers consider moral hazard as a microeconomic foundation of the financial constraint. Ai, Li, and Li (2017) also study the financial constraint and the implied investment-cash-flow sensitivity based on a dynamic moral hazard model. They focus on the failure of traditional Q theory and the negative dependence of the investment-cash-flow sensitivity on firm size and age. In contrast, this paper observes the factors of the production process that affect the relation between the investment-cash-flow sensitivity and the financial constraint.

In terms of modeling the financial constraint, this paper relates to Clementi and Hopenhayn (2006). Both papers study a financial constraint implied by a dynamic moral hazard problem which restricts the scale of the production financed by the outside investor.[3] However, they pay attention to the properties of firm dynamics implied by the financial con-

[3] Albuquerque and Hopenhayn (2004) model the financial constraint in a similar way, but their constraint is a result of the limited commitment in financial contracts.

straint. This paper studies how the investment-cash-flow sensitivity depends on the financial constraint. Furthermore, this paper emphasizes the quantitative implications of the model and shows how the calibrated benchmark model matches the data.

The rest of the paper is organized as follows. In Section 2, I lay out the model, and in Section 3, I characterize the optimal contract. In Section 4, I show how investment-cash-flow sensitivity depends on the endogenous financial constraint; Section 5 interprets the investment-cash-flow sensitivity regressions based on the simulation data. Section 6 shows how I calibrate the benchmark model and how it fits the data, and Section 7 concludes.

2 The model

The model is an extension of DeMarzo and Sannikov (2006). The time horizon of the model is $[0, \infty)$. A unit measure of risk-neutral entrepreneurs arrives at the economy per unit of time. Each entrepreneur is endowed with a technology that allows him to produce consumption goods from capital over a long time horizon by establishing a firm. Let us consider a firm established at time zero. The productivity of the firm at $t \geq 0$ is

$$Z_t = \exp(\mu t),$$

with μ being the productivity growth rate. As in DeMarzo and Sannikov (2006), let Y_t be the quantity of cash flows that the firm generates up to time t. Given a sequence of capital inputs, $\{K_t\}$, the cash flow rate at time t is given by

$$dY_t = Z_t^{1-\alpha} K_t^{\alpha} + K_t \sigma dB_t. \tag{1}$$

Here, $\alpha \in (0, 1)$ is the capital share of the production technology, $\{B_t\}$ is a standard Brownian motion characterizing the idiosyncratic cash flow shocks, and $\sigma > 0$ is the rate of volatility. Each unit of capital inputs requires a user's cost, $r + \delta + \kappa$, per unit of time, where $r > 0$, $\delta > 0$, and $\kappa > 0$ are the interest rate, the capital depreciation rate, and the death rate of existing entrepreneurs, respectively. In this model, I assume that entrepreneurs are hit by death shocks independently with a fixed Poisson rate $\kappa > 0$. Upon the death shock, the entrepreneur and the firm exit the economy. I denote the Poisson time of the death shock by τ.[4]

[4]Because of the birth and death of the firms, the economy has a steady state.

The entrepreneur does not have initial wealth and asks an investor to finance the cost of capital. The investor provides financing by offering a lending contract, $(\{C_t\}, \{K_t\}, T)$. Specifically, C_t is the total transfers that the investor pays to the entrepreneur up to time t, and dC_t is the instantaneous rate of transfer, which has to be non-negative because the entrepreneur is protected by limited liability. The term K_t is the quantity of capital financed by the investor at t, and T is the time to liquidate the firm. All three terms depend on the entire history. Under the contract, the entrepreneur reports and hands over the firm's cash flows to the investor. Upon liquidation, the firm does not generate residual values.[5]

Moral hazard arises because the cash flow shocks are not observable to the investor. Therefore, the entrepreneur could misreport and secretly divert cash flows to increase consumption. Under the contract, if the entrepreneur diverts D_t for $t \geq 0$, his expected utility is

$$E_0\left[\int_0^{\tau\wedge T} e^{-\beta t}\left(D_t dt + dC_t\right)\right],$$

where $\beta > 0$ is his discount rate, and E_0 is the time zero expectation operator.[6] On the other hand, the investor's expected payoff is

$$E_0\left[\int_0^{\tau\wedge T} e^{-rt}\left[\left(Z_t^{1-\alpha}K_t^{\alpha} - (r+\delta+\kappa)K_t - D_t\right)dt - dC_t\right]\right]. \quad (2)$$

I assume that $\beta > r$ so that the investor is more patient. Since there is no upper bound on the amount of the cash flows that the entrepreneur could divert, I focus on the incentive-compatible contract under which he is induced to truthfully report the cash flows.[7] The investor designs the optimal incentive-compatible lending contract to maximize her expected payoff, (2), and promises the entrepreneur an initial expected utility U_0. To guarantee that the firm value to the investor is finite, I make the following assumption.

Assumption 1. $r + \kappa > \mu$.

Clearly, the larger the parameter σ, the harder it is to infer the actual cash flows from the entrepreneur's reports. Therefore, σ indicates the severity of the moral hazard problem. Moreover, given σ, the volatility is proportional to the capital stock K_t. Intuitively, when

[5]This assumption is not essential to the key results of the paper.

[6]Obviously, the probability basis of this expectation depends on the entrepreneur's diversion behavior.

[7]See DeMarzo and Sannikov (2006) for the argument about the optimality of doing so.

the firm is larger and the entrepreneur oversees a greater quantity of cash flows, more intense incentive provisions are required to deter hidden diversions. Since the model allows long-run growth of the firms, this assumption prevents firms from growing out of the moral hazard problem.

3 The optimal contract

3.1 Normalization and incentive compatibility

By following the literature, I define the entrepreneur's continuation utility

$$U_t = E_t \left[\int_t^{\tau \wedge T} e^{-\beta(s-t)} dC_s \right] \text{ for } t \in [0, \tau \wedge T]$$

as one of the state variables. Let $V(Z, U)$ be the value function of the investor's maximization problem. Given the homogeneity of this problem, the value function satisfies

$$V(Z, U) = Zv\left(\frac{U}{Z}\right),$$

where $v(u)$ is the normalized value function and $u = \frac{U}{Z}$ is the entrepreneur's normalized continuation utility, which is the ratio of his future payments to the scale of production and can be interpreted as the entrepreneur's stake in the firm. Accordingly, I define the normalized capital input and transfer payment to the entrepreneur, $k_t = \frac{K_t}{Z_t}$ and $dc_t = \frac{dC_t}{Z_t}$, respectively. The Martingale representation theorem implies the following law of motion of u_t.

Lemma 1. *Suppose the contract* $(\{C_t\}, \{K_t\}, T)$ *is incentive compatible and the entrepreneur's normalized continuation utility satisfies*

$$du_t = u_t(\beta + \kappa - \mu)\, dt - dc_t + g_t \sigma dB_t. \tag{3}$$

Here $\{g_t\}$ *is a predictable process such that* $\{g_t Z_t\}$ *is square integrable.*

Proof. See Appendix A. □

Since dB_t represents the noise in the entrepreneur's cash-flow reports, g_t is the sensitivity of his continuation utility with respect to his reports. Therefore, g_t determines the entrepreneur's incentives to truthfully report cash flows or not report them. Hence, we have the following incentive compatibility condition.

Lemma 2. *A contract* $(\{C_t\}, \{K_t\}, T)$ *is incentive compatible if and only if*

$$g_t \geq k_t \text{ for all } t \in [0, \tau \wedge T]. \tag{4}$$

Proof. See Appendix B. □

According to (4), g_t has to be no lower than the normalized working capital level, k_t. Intuitively, a greater pay-performance sensitivity is required to deter hidden diversions when the entrepreneur oversees a larger scale of production. In fact, this condition is binding under the optimal contract because the unnecessary exposure of u_t to the cash flow shocks reduces efficiency, as I prove in Proposition 2 in Appendix C.

3.2 A brief characterization of the optimal contract

In this subsection, I briefly describe the optimal contract and the normalized value function, $v(u)$, and leave the detailed discussions to Appendix C. There is an upper bound $\hat{u}$, such that, under the optimal contract, if $u_t \in [0, \hat{u}]$, the investor does not pay the entrepreneur and u_t evolves according to

$$du_t = u_t (\beta + \kappa - \mu) dt + k_t \sigma dB_t. \tag{5}$$

Once u_t reaches zero, the firm is liquidated, namely, $T = \inf\{t : u_t = 0\}$. Given any history, if $u_t \geq \hat{u}$, a lump-sum transfer, $dc_t = u_t - \hat{u}$, is paid to the entrepreneur immediately so that the transfers that need to be paid in the future decrease and u_t reflects back to $\hat{u}$. Over $[0, \hat{u}]$, $v(u)$ is strictly concave and satisfies

$$0 = \max_{k \geq 0} k^\alpha - (r + \delta + \kappa) k - (r + \kappa - \mu) v(u) + (\beta + \kappa - \mu) u v'(u) + \frac{1}{2} v''(u) \sigma^2 k^2. \tag{6}$$

Over $[\hat{u}, \infty)$, $v'(u) = -1$. The optimal capital input $k(u)$ maximizes the objective function on the right-hand side of (6).

3.3 The agency cost and endogenous financial constraint

In this subsection, I show how the moral hazard problem implies a financial constraint on capital financing. Recall that the entrepreneur's stake, u, is the expected present value of the payments that the entrepreneur is going to receive under the lending contract. Hence, to deter hidden diversions, this stake has to be sensitive to the cash flows reported by him so that it increases upon positive shocks and decreases upon negative ones. Since the entrepreneur is protected by limited liability, as in DeMarzo and Sannikov (2006),[8] liquidation has to be used as ultimate punishment when u decreases to zero after a sequence of negative cash flow shocks,[9] even though liquidation is inefficient due to the forgone future income. When deciding the level of capital to finance, the investor needs to take into account the possibility of liquidation. According to Lemma 2, when the firm is producing on a larger scale, stronger incentive provisions are needed to deter diversions, which increase the volatility of u. Hence, a higher level of capital would raise the probability of liquidation, and then an agency cost of financing capital arises, which endogenously implies a financial constraint. This financial constraint forces the investor to choose a capital level strictly lower than the first-best one.[10] Obviously, as the level of u increases, the risk of liquidation diminishes, and the agency cost decreases, allowing the investor to finance a higher and more efficient level of capital.

Technically, if there were no moral hazard, the investor would always choose the first-best level of k that maximizes the operating profit,

$$k^{\alpha} - (r + \delta + \kappa)\, k.$$

With moral hazard, according to (6) it maximizes

$$k^{\alpha} - (r + \delta + \kappa)\, k + \frac{1}{2} v''(u) \sigma^2 k^2.$$

The last term in the expression above stands for the agency cost of incentive provisions. Since $v(u)$ is strictly concave, the larger the $|v''(u)|$, the more k is restricted from its efficient level, and the tighter the financial constraint.[11]

[8] See He (2009) and DeMarzo, Fishman, He, and Wang (2012) for the argument with a geometric Brownian motion cash-flow process.

[9] Due to the limited liability of the entrepreneur, negative transfer payments are not feasible. Therefore, the only way to creditably make his stake zero is to liquidate the firm and terminate the contract, because, otherwise, the entrepreneur can always divert cash flows as long as the firm is being financed with capital and producing.

[10] We discuss the first-best case in Appendix D.

[11] In fact, the concavity of $v(u)$ is the "implied risk aversion" in my model, because the volatility of u_t raises the probability of the liquidation and reduces efficiency.

I plot the $|v''(u)|$ and the normalized capital input function, $k(u)$, in the upper and lower panels of Figure 1, respectively. As seen in the diagram, $|v''(u)|$ monotonically decreases with u over $[0, \hat{u}]$. Obviously, the probability of liquidation and the agency cost diminish as u increases. Thus, the capital input increases with u as the financial constraint is relaxed and reaches its first-best level, k^*, at $\hat{u}$ where the agency cost vanishes. Therefore, to alleviate the agency cost and relax the financial constraint, the transfer payments promised to the entrepreneur are deferred even though he is less patient, so that u_t drift up with rate $\beta + \kappa - \mu > 0$,[12] and the investor can choose a higher and more efficient capital level to finance.

I assume that the investor has full bargaining power when offering the contract. Therefore, she chooses the initial continuation utility U_0 to maximize the firm value to her; namely, the initial normalized continuation utility, u_0, satisfies $u_0 = \arg\max_{u \in [0,\hat{u}]} v(u)$.

3.4 The investment-cash-flow sensitivity

In this subsection, I derive a simple expression for the investment-cash-flow sensitivity under the optimal contract. Instantaneously, the investment-to-capital ratio is $\frac{dK_t}{K_t} + \delta dt$. Given (5), Itô's lemma implies that, for all $t \in [0, \tau \wedge T]$,

$$\frac{dK_t}{K_t} = \left[\mu - \delta + \frac{k'(u_t)(\beta + \kappa - \mu) u_t + \frac{1}{2} k''(u_t) \sigma^2 k(u_t)^2}{k(u_t)}\right] dt + k'(u_t) \sigma dB_t. \tag{7}$$

On the other hand, (1) implies

$$\frac{dY_t}{K_t} = k(u_t)^{\alpha-1} dt + \sigma dB_t. \tag{8}$$

Hence, (7) and (8) straightforwardly imply the following result.

Proposition 1. *Under the optimal contract, the instantaneous investment-cash-flow sensitivity is*

$$\frac{Cov\left(\frac{dK_t}{K_t}, \frac{dY_t}{K_t}\right)}{Var\left(\frac{dY_t}{dK_t}\right)} = k'(u_t). \tag{9}$$

If there were no moral hazard, the normalized capital input would be time invariant so that $\frac{dK_t}{K_t} = dZ_t = \mu dt$. Therefore, we have the following corollary.

[12] See the law of motion of u_t under the optimal contract, (5).

Corollary 1. *In the first-best case, the investment-cash-flow sensitivity is zero.*

With moral hazard, positive cash flow shocks relax the constraint, enabling the investor to finance more capital, whereas negative shocks tighten the constraint, forcing the investor to reduce capital. Consequently, the model endogenously implies a positive investment-cash-flow sensitivity.

4 Relationship between investment-cash-flow sensitivity and financial constraint

Given the expression for the investment-cash-flow sensitivity in Proposition 1, I study how the returns to scale in capital of the production technology, α, and the severity of the moral hazard problem, σ, affect the relationship between the investment-cash-flow sensitivity and the financial constraint. Notice that the financial constraint directly restricts the firm's capital financing. The tighter the financial constraint, the further the normalized capital level, $k(u)$, is below the first-best one, k^*. Therefore, the deviation of $k(u)$ from k^*, $\left(k^* - k(u)\right)/k^*$, is a straightforward measure of the tightness of the constraint, which I use in the interpretation. Specifically, based on the calibrated benchmark model introduced in Section 6, I take three different levels of α and σ, calculate the optimal contract for each, and plot $k'(u)$ as a function of the deviation of capital financing in the left and right panels of Figure 2, respectively.

As seen in the left panel, when the capital share, α, is close to one, the dependence of the investment-cash-flow sensitivity on the financial constraint significantly decreases. As α decreases, the significance of this pattern diminishes, and the dependence increases in some region if α is too small. To understand this phenomenon, notice that, in a less constrained firm, a positive cash-flow shock, dB_t, induces a larger increase in u_t and a greater relaxation of the financial constraint because of the positive dependence of pay-performance sensitivity on the capital level (incentive constraint (4)). If the production technology exhibits almost constant returns to scale, the marginal product of capital does not diminish significantly; thus, a greater relaxation of the financial constraint implies a larger increase in capital. Therefore, a less constrained firm exhibits a greater investment-cash-flow sensitivity. It is easy to see that if the marginal product of capital diminishes too quickly when the firm deploys more capital, the pattern can be the opposite, as shown in the diagram.

According to the right panel of Figure 2, if the contract is subject to a severe moral hazard problem, the magnitude of the investment-cash-flow sensitivity decreases with the tightness of the financial constraint. If the severity decreases, the pattern is reversed in some region. The tightness of the financial constraint is determined by the probability of the liquidation. According to the law of motion of u_t (equation (5)), the liquidation probability over a unit length of time is approximately the left tail, cut off at zero, of a normal distribution with mean u_t and standard deviation $\sigma k(u_t)$. Given the bell shape of the normal distribution, a large σ implies that the liquidation probability is still sensitive to the changes in u_t even if u_t is at a high level (less constrained). Moreover, the incentive provision is multiplicative in $k(u_t)$. Therefore a unit size of positive cash flow shock induces a larger increase in u_t and $k(u_t)$. As a result, the investment-cash-flow sensitivity increases with u_t and decreases with the tightness of the financial constraint. However, if σ is relatively small, the liquidation probability is sensitive to cash flow shocks only if u_t is close to zero (severely constrained). Hence, as u_t goes up, it quickly becomes insensitive to the cash flow shocks, and the investment-cash-flow sensitivity diminishes, as indicated by the solid curve in the right panel of Figure 2.

5 Investment-cash-flow sensitivity regressions

By using the simulation data, I regress the investment-to-capital ratio on Tobin's Q and the cash-flow-to-capital ratio to study how the coefficient on the cash-flow-to-capital ratio depends on the following three firm characteristics: (1) the tightness of the financial constraint, (2) the firm size, and (3) the firm age. I particularly want to see how these dependence are affected by the parameters, α and σ. Let the integer $\bar{t} = 0, 1, 2, ...$, be the index of year, and let $K_{\bar{t}}$ and $Q_{\bar{t}}$ be the capital input level and Tobin's Q at the beginning of year $\bar{t}$, respectively. The investment-to-capital ratio over this year is

$$i_{\bar{t}} = \ln(K_{\bar{t}+1}) - \ln(K_{\bar{t}}) + \delta,$$

and the cash-flow-to-capital ratio is $cf_{\bar{t}} = \frac{CF_{\bar{t}}}{K_{\bar{t}}}$ with

$$CF_{\bar{t}} = \int_{\bar{t}}^{\bar{t}+1} dY_t = \int_{\bar{t}}^{\bar{t}+1} \left[K_t^{\alpha} Z_t^{1-\alpha} dt + \sigma K_t dB_t\right].$$

As in Section 4, based on the benchmark parameter values calibrated in Section 6, I choose three different values of α and σ, respectively. For each specification of the parameter values, I simulate the optimal contract,[13] and I divide firm samples into five groups with an equal number of samples in each group according to each of the three firm characteristics listed above. I run the investment-cash-flow sensitivity regression in each group. As in the literature, the repression equation is[14]

$$i_{\bar{t}} = \beta_0 + \beta_Q Q_{\bar{t}} + \beta_{CF} cf_{\bar{t}} + \varepsilon.$$

In Table 1, for different values of α and σ, I report the regression coefficients on the cash-flow rate, β_{CF}, in different groups subject to different levels of the financial constraint.[15] As in Section 4, the tightness of the financial constraint is measured by the deviation of capital financing from its first-best level.

As seen in Table 2, when α or σ is small, less constrained groups generally exhibit smaller coefficients. As α or σ goes up, the pattern becomes the opposite. Under my framework, firm size and age can be proxies for the financial constraint. Intuitively, when a new firm is established, it is subject to a tight financial constraint because the entrepreneur's stake in the firm is small and the capital level is low. As the firm grows larger and older, the entrepreneur's stake drifts up (equation (5)) and the financial constraint is relaxed. As a result, firm size and age decrease with the tightness of the financial constraint so that large and old firms are less constrained.

In Tables 2 and 3, firm samples are divided into different groups according to their size and age, respectively. As expected, if the parameter, α or σ, is small, the regression coefficient on the cash flow rate significantly decreases with size and age. The significance diminishes as the parameter value increases, and the pattern is reversed if the parameter value is too

[13]I simulate an economy consisting of $20,000$ firm positions for 350 years and collect the firm-year samples in the last 50 years making sure that the cross-sectional distribution of the firms over the state space is time invariant. For each simulation, I obtained approximately $900,000$ samples.

[14]In the model, there is no firm or year fixed effect. Notice that the coefficients are significantly larger than that documented in the empirical literature. There could be some additional random shocks in cash flows in reality that are not modeled in my framework.

[15]All the regression coefficients for the simulation data are significant, so I only report their values in the tables.

large.

6 Calibration of the benchmark model

I show how I calibrate the benchmark parameter values of the model and that the benchmark model quantitatively accounts for the negative dependences of the investment-to-capital ratio and Tobin's Q on firm size and age.

To calibrate the benchmark model, I first choose the interest rate $r = 4\%$ to be the average return of risky and risk-free assets in the U.S. postwar period, and the capital depreciation rate $\delta = 10\%$, the depreciation rate documented in the real business cycle literature.[16] The rest of the parameters are chosen by matching the moments estimated from the manufacturing firms in the Compustat data set for the period 1967 to 2015, which Table 4 summarizes.

Specifically, I choose the parameter of the returns to scale of capital $\alpha = 0.8$ to match the median cash-flow-to-capital ratio in the data, which is 35%. The death rate $\kappa = 0.05$ per annum is the average death rate of the firms in the data. I set the productivity growth rate $\mu = 0.049$ to match the median growth rate of the old firms,[17] which is 7.6% and $\beta = 0.08$ to match the median growth rate of the entire data set, which is 8.8%. I choose the volatility of the cash flow shocks, $\sigma = 0.35$, to match the standard deviation of the cash-flow-to-capital ratio in the data, which is 44.2%.

For both the Compustat data and the simulation data, I divide firm-year samples into five groups according to their initial sizes and ages in a year, respectively, with an equal number of samples in each group. Then, I evaluate the median investment-to-capital ratio in each group and report them in Table 5. The firms in the groups 1 to 5 are from small to large or from young to old. As seen in the table, the investment-to-capital ratio decreases with size and age and that pattern is consistent with the data. In fact, the negative dependence of the investment-to-capital ratio on size and age has been documented in empirical studies; for example, see Evans (1987) and Hall (1987). The economic mechanism behind this pattern in the model is clear. In early stage of a firm's life cycle, u_t is at a low level and the firm is subject to a tight financial constraint, which restricts the firm's investment. In this

[16] For example, see Mehra and Prescott (1985).

[17] I define old firms to be the those that are older than the median age.

early stage, the firm's growth and investment are driven by the relaxation of the financial constraint and the progress of productivity. Once the agency cost vanishes and the financial constraint is relaxed, the firm matures, and its growth and investment are driven only by productivity growth. Consequently, large and old firms grow slower than their small and young counterparts.

Consistent with the data, Tobin's Q decreases with size and age. Intuitively, in my model, Tobin's Q is largely determined by the marginal product of capital. Large and old firms are less constrained so that they deploy higher levels of capital and thus have lower marginal products of capital.

7 Conclusion

I propose a theoretical framework under which a moral hazard problem endogenously induces a financial constraint on investment so that investment responds to cash flow shocks. I show that, under this framework, whether the investment-cash-flow sensitivity increases or decreases with the tightness of the financial constraint substantially depends on the returns to scale in capital of the production technology and the severity of the moral hazard problem. If the technology exhibits almost constant returns to scale or the production is subject to a severe moral hazard problem, investment-cash-flow sensitivity is negatively correlated with the financial constraint; otherwise, the correlation could be positive. This result emphasizes the controls for the production technology and the monitoring structure of firm organization in future empirical studies. Moreover, the calibrated benchmark model can replicate the negative dependence of the investment-to-capital ratio and Tobin's Q on firm size and age that is observed in the data.

Appendix

A Proof of Lemma 1

Let Υ_t be the time-t conditional expectation of the entrepreneur's total utility under the contract. Then I have

$$\Upsilon = E_t\left[\int_0^T e^{-(\beta+\kappa)t}dC_t\right] = \int_0^t e^{-(\beta+\kappa)s}dC_s + e^{-(\beta+\kappa)t}U_t. \tag{A.10}$$

So $\{\Upsilon_t\}$ is an adapted martingale and the Martingale representation theorem implies

$$d\Upsilon_t = e^{-(\beta+\kappa)t}G_t\sigma dB_t \tag{A.11}$$

with $\{G_t\}$ being a predictable and square integrable process. Equation (A.10) and (A.11) implies the law of motion U_t and then (3) according to Itô's lemma if I define $g_t = \frac{G_t}{Z_t}$.

B Proof of Lemma 2

I show the following incentive compatibility condition which is equivalent to (4).

$$G_t \geq K_t \text{ for all } t \in [0, \tau \wedge T]. \tag{A.12}$$

Given any diversion policy of the entrepreneur, $\{D_t\}$, define $\left\{B_t^D\right\}$ such that

$$dB_t^D = \frac{dY_t - \left(Z^{1-\alpha}K_t^\alpha - D_t\right)dt}{\sigma K_t} \text{ for all } t \in [0, \tau \wedge T]. \tag{A.13}$$

Given any $t \in [0, \tau \wedge T]$, I define

$$\mathcal{G}_t^D = \int_0^t e^{-(\beta+\kappa)s}\left(D_s ds + dC_s\right) + e^{-(\beta+\kappa)t}U_t.$$

Intuitively, $\mathcal{G}_t$ is the time-t conditional expected utility of the manager if he diverts cash flows according to $\{D_t\}$ until t and then stops doing so. Clearly, $\mathcal{G}_0^D = U_0$. Condition (A.13) implies

$$e^{-(\beta+\kappa)t}d\mathcal{G}_t^D = D_t\left(1 - \frac{G_t}{K_t}\right)dt + G_t\sigma dB_t^D.$$

The second term on the right hand side is a martingale under the diversion policy $\{D_t\}$. The entrepreneur is not willing to divert cash flows if and only if $\left\{\mathcal{G}_t^D\right\}$ is a super martingale, and if and only if (A.12) is satisfied.

C Characterization of the optimal contract

I discuss the normalized value function, $v(u)$, and the optimal contract in more details which is extended from DeMarzo and Sannikov (2006) and He (2009). The law of motion of u_t, (3), implies that $v(u)$ satisfies the following HJB differential equation

$$0 = \max_{k,g \geq k,dc} \quad k^{\alpha} - (r + \delta + \kappa) k - (r + \kappa - \mu) v(u) + (\beta + \kappa - \mu) uv'(u) \\ + \frac{1}{2} v''(u) g^2 \sigma^2 - (1 + v'(u)) dc \tag{A.14}$$

The investor can always lower the entrepreneur's normalized continuation utility instantaneously by delivering him a lump-sum transfer. Therefore,

$$v(u) \geq v(u - dc) - dc \text{ for all } dc > 0 \text{ and } v'(u) \geq 1.$$

Proposition 2. *The optimal contract promising the manager initial continuation utility* $U_0 = u_0 Z_0 = u_0$ *takes the following form. For* $t \leq \tau \wedge T$*, when* $u_t \in (0, \hat{u})$*,* u_t *evolves according to* (5)*,* $dc_t = 0$*, and* $k_t = k(u_t)$*, which is the maximizer of the right-hand side of* (6) *with level* u_t*; when* $u_t > \hat{u}$*,* $dc_t = u_t - \hat{u}$*, reflecting* u_t *back to* $\hat{u}$*. The firm is liquidated at time* T*, the time when* u_t *reaches zero. The normalized value function,* v*, satisfies* (6) *over* $[0, \hat{u}]$ *with boundary conditions,* $v(0) = 0$*,* $v'(\hat{u}) = -1$*,* $v''(\hat{u}) = 0$*, and*

$$v(\hat{u}) = \frac{\pi^*}{r + \kappa - \mu} - \frac{\beta + \kappa - \mu}{r + \kappa - \mu} \hat{u} \text{ with } \pi^* = (1 - \alpha) \left(\frac{\alpha}{r + \delta + \kappa} \right)^{\frac{\alpha}{1-\alpha}}. \tag{A.15}$$

For $u \geq \hat{u}$*,* $v(u) = v(\hat{u}) - (u - \hat{u})$*. Moreover,* v *is strictly concave over* $[0, \hat{u}]$*.*

The proof is divided into the following three lemmas. In the first lemma I show the concavity of the normalized value function.

Lemma 3. *The normalized value function* v *satisfying HJB* (6)*, the boundary conditions* $v'(\hat{u}) = -1$*,* (A.15)*, and* $v''(\hat{u}) = 0$ *is concave over* $[0, \hat{u}]$*.*

Proof. I divide both hand sides of (6) with respect to u based on the Envelope theorem and obtain

$$(\beta - r) v'(u) + (\beta + \kappa - \mu) uv''(u) + \frac{1}{2} v''' k(u)^2 \sigma^2 = 0. \tag{A.16}$$

which implies, given the boundary conditions at $\hat{u}$,

$$v'''(\hat{u}) = \frac{2(\beta - r)}{k(\hat{u})^2 \sigma^2} > 0.$$

So for any real number $\epsilon > 0$ which is small enough, $v''(\hat{u} - \epsilon) < 0$. Suppose that $v(u)$ is not concave and $\tilde{u}$ is the largest real number over $[0, \hat{u}]$ such that $v''(\tilde{u}) = 0$. Then (6) implies that $k(\tilde{u}) = k^*$ and

$$v(\tilde{u}) = \frac{\pi^*}{\beta + \kappa - \mu} + \frac{\beta + \kappa - \mu}{r + \kappa - \mu} \tilde{u} v'(\tilde{u}). \tag{A.17}$$

Since $v'(u) > -1$ for all $u \in (\tilde{u}, \hat{u})$, (A.17) contradicts (A.15) and I have the desired result. $\square$

Verification of $v(u)$ and the described contract are divided into Lemmas 4 and 5. Lemma 4 shows that the payoffs indicated by $v(u)$ is achievable under the described contract.

Lemma 4. *Under the contract described in Proposition 2, the normalized firm value is characterized by $v(u)$.*

Proof. Clearly, under the described contract, u_t follows (5). Now suppose that $U_0 = u_0 \in [0, \hat{u})$. I define

$$\Psi_t = \int_0^t e^{-(r+\kappa)s} \left[Z_s^{\alpha} K_s^{1-\alpha} - (r + \delta + \kappa) K_s - dC_s \right] + e^{-(r+\kappa)t} Z_t v(u_t) \text{ for any } t \in [0, \tau \wedge T]$$

and, obviously, $\Psi_0 = Z_0 v(u_0) = v(u_0)$. Then, according to Itô's lemma,

$$e^{-(r+\kappa)t} d\Psi_t = Z_t \begin{bmatrix} k_t^{\alpha} - (r + \delta + \kappa) k_t - (r + \kappa - \mu) v(u_t) + (\beta + \kappa - \mu) u_t v'(u_t) \\ + \frac{1}{2} v''(u_t) k_t^2 \sigma^2 - (1 + v'(u_t)) dc_t + v'(u_t) k_t \sigma dB_t \end{bmatrix}. \tag{A.18}$$

Under the contract, $dc_t \neq 0$ if and only if $v'(u_t) = -1$ and k_t is the optimal solution of the maximization problem on the right hand side of (6). So (A.18) is simplified to

$$e^{-(r+\kappa)t} d\Psi_t = Z_t v'(u_t) k_t \sigma dB_t,$$

and then $\{\Psi_t\}$ is an adapted martingale. Therefore, the expected payoff under the contract is $E_0[\Psi_\infty] = \Psi_0 = v(u_0)$ and I have the desired result. $\square$

Lemma 5. *Any incentive compatible contract promising the entrepreneur expected utility $U_0 = Z_0 u_0 = u_0$ cannot generate an expected payoff larger than $Z_0 v(u_0) = v(u_0)$.*

Proof. Because of the limited liability of the entrepreneur, the firm has to be liquidated when u_t hits zero. I denote the hitting time by T_0 which could be infinite. Let $\left(\left\{\hat{C}_t\right\}, \left\{\hat{K}_t\right\}, \hat{T}\right)$ be an alternative incentive compatible contract promising the entrepreneur initial expected utility u_0 and I use hatted letters denote corresponding terms under this contract. For any $t \in \left[0, \tau \wedge \hat{T} \wedge T_0\right]$, define

$$\hat{\Psi}_t = \int_0^t e^{-(r+\kappa)s}\left[\left(Z_s^\alpha \hat{K}_s^{1-\alpha} - (r+\delta+\kappa)\hat{K}_s\right)ds - d\hat{C}_s\right] + e^{-(r+\kappa)t} Z_t v\left(\hat{u}_t\right).$$

Obviously, $\hat{\Psi}_0 = Z_0 v(u_0) = v(u_0)$. Then

$$e^{-(r+\kappa)t} d\hat{\Psi}_t = Z_t \begin{bmatrix} \hat{k}_t^\alpha - (r+\delta+\kappa)\hat{k}_t - (r+\kappa-\mu)v(\hat{u}_t) + (\beta+\kappa-\mu)\hat{u}_t v'(\hat{u}_t) \\ +\frac{1}{2}v''(\hat{u}_t)\hat{g}_t^2\sigma^2 - (1+v'(\hat{u}_t))d\hat{c}_t + v'(\hat{u}_t)\hat{g}_t\sigma dB_t \end{bmatrix}.$$

Concavity of $v(u)$ (Lemma 3), the fact that $v' > -1$, and the HJB (6) imply that $\left\{\hat{\Psi}_t\right\}$ is a super martingale. Therefore the expected payoff of the investor under the alternative contract is

$$E_0\left[\hat{\Psi}_{\tau\wedge\hat{T}\wedge T_0}\right] \leq E_0\left[\hat{\Psi}_0\right] = Z_0 v(u_0) = v(u_0).$$

□

D The first-best case

In this section, I consider the first-best case in which there is no moral hazard. Obviously, in this case, the firm is never liquidated and $K_t = k^* Z_t$ for $t \leq \tau$ with

$$k^* = \arg\max_k k^\alpha - (r+\delta+\kappa)k = \left(\frac{\alpha}{r+\delta+\kappa}\right)^{\frac{1}{1-\alpha}},$$

and the instantaneous expected operating profit rate is $\pi^* Z_t$ with

$$\pi^* = (1-\alpha)\left(\frac{\alpha}{r+\delta+\kappa}\right)^{\frac{\alpha}{1-\alpha}}.$$

Therefore, the net present value of the total cash flows generated by the firm is $Z_0\frac{\pi^*}{r+\kappa-\mu} = \frac{\pi^*}{r+\kappa-\mu}$. Since the investor is more patient, it is optimal to pay off all the transfers promised to the entrepreneur at $t=0$. Therefore, the first-best normalized value function $v^{FB}(u)$ satisfies

$$v^{FB}(u) = \frac{\pi^*}{r+\kappa-\mu} \text{ for all } u \geq 0.$$

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In Table 6, I report the median Tobin's Q in each size or age group.

Figure 1.

Agency Cost and Optimal Capital Input.

In the upper panel I plot the absolute value of the second-order-derivative of v as a function of the entrepreneur's normalized continuation utility, u. In this model, $|v''(u)|$ reflects the agency cost. In the bottom panel I plot the normalized capital input, k, as a function of u. The agency cost diminishes at level $\hat{u}$ and the normalized capital input reaches its first-best level.

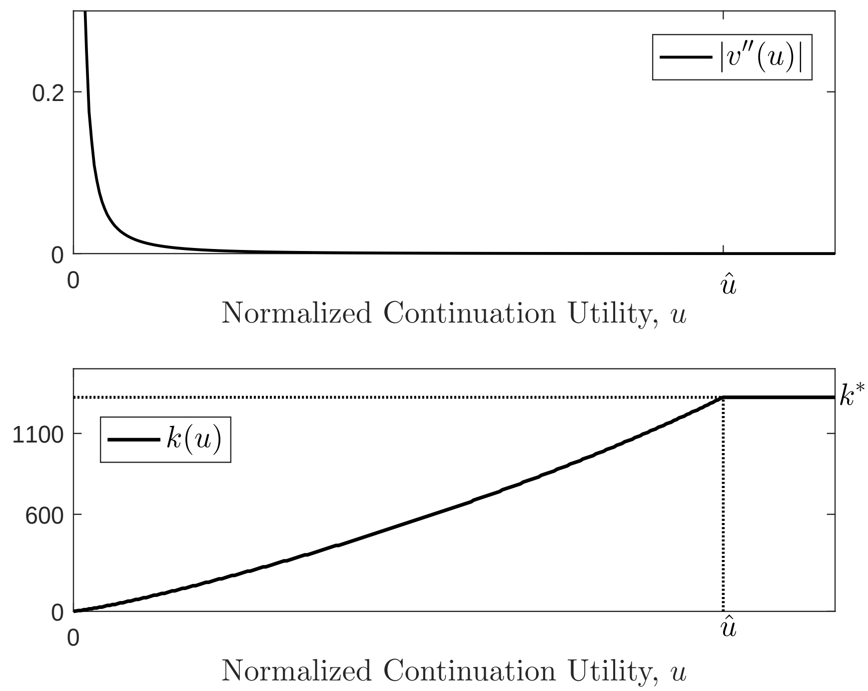

Figure 2.

Dependence of Investment-Cash-Flow Sensitivity on Tightness of Financial Constraint.

In both panels, I plot the instantaneous investment-cash-flow sensitivity, defined in equation (9), as a function of the deviation of capital financing, which indicates the tightness of the financial constraint. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α and σ, and plot the diagrams in the left and right panels respectively.

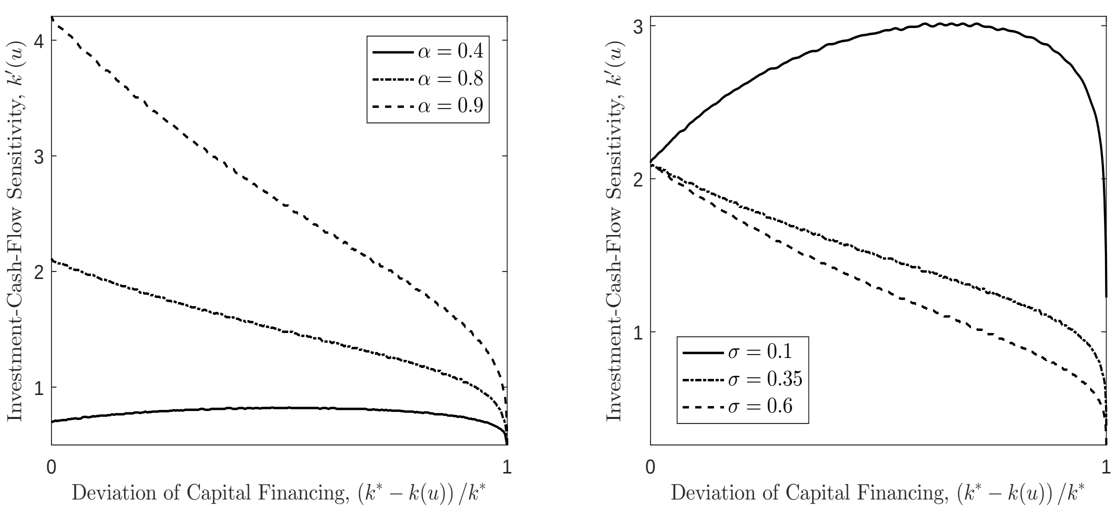

Table 1.

Dependence of investment-cash-flow sensitivity regression coefficient on tightness of the financial constraint with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the financial constraints the group members subject to relaxes. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Constrained	0.68	0.84	0.88	2.22	0.83	0.51
2	0.76	0.95	1.04	2.35	0.95	0.60
3	0.76	1.04	1.13	2.33	1.04	0.66
4	0.63	1.17	1.27	1.96	1.17	0.77
Unconstrained	0.36	1.05	1.54	1.17	1.05	0.87

Table 2.

Dependence of investment-cash-flow sensitivity regression coefficient on firm size with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the size of the group members increases. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Small	0.70	0.88	0.93	2.26	0.88	0.54
2	0.74	0.97	1.05	2.32	0.97	0.61
3	0.68	1.06	1.14	2.14	1.06	0.68
4	0.58	1.12	1.30	1.83	1.13	0.78
Large	0.57	1.04	1.46	1.77	1.04	0.81

Table 3.

Dependence of investment-cash-flow sensitivity regression coefficient on age with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the age of the group members increases. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Young	0.73	1.00	1.11	2.29	1.00	0.65
2	0.67	1.02	1.17	2.14	1.02	0.68
3	0.64	1.02	1.20	2.05	1.02	0.69
4	0.63	1.02	1.22	2.00	1.02	0.70
Old	0.62	1.02	1.23	1.99	1.02	0.71

Table 4.

Calibrated parameter value and targeted moments.

Model-specific parameters		Value	Targeted Moments
α	Returns to scale parameter	0.800	Median cash flow rate of Computat firms
κ	Average death rate	0.050	Average death rate of Compustat firms
μ	Productivity growth rate	0.049	Median growth of old Compustat firms
β	Discount rate	0.08	Median growth rate of Compustat firms
σ	Volatility of unobservable shock	0.35	Volatility of cash flow rate of Compustat firms

Table 5.

Investment-to-capital ratio by size and age.

Firm samples observed in the data and generated in the simulation are respectively divided into five groups with equal number of samples in each group according to their size (column 2 and 3) and age (column 4 and 5). From group 1 to 5, size or age increases. I calculate the average investment-to-capital ratio in each group. In both the data and the simulation, the investment-to-capital ratio decreases with size and age. The model simulation is based on on the calibrated benchmark model introduced in Section 6.

	Size Group		Age Group	
Group No.	Data	Model	Data	Model
1	0.21	0.20	0.21	0.19
2	0.20	0.19	0.20	0.18
3	0.19	0.19	0.19	0.17
4	0.18	0.17	0.19	0.16
5	0.18	0.11	0.18	0.15

Table 6.

Tobin's Q by size and age.

Firm samples observed in the data and generated in the simulation are respectively divided into five groups with equal number of samples in each group according to their size (column 2 and 3) and age (column 4 and 5). From group 1 to 5, size or age increases. I calculate the average Tobin's Q in each group. In both the data and the simulation, the Tobin's Q decreases with size and age. The model simulation is based on on the calibrated benchmark model introduced in Section 6.

	Size Group		Age Group	
Group No.	Data	Model	Data	Model
1	3.03	6.21	2.89	3.24
2	3.00	3.33	2.78	2.77
3	2.24	2.21	2.36	2.31
4	2.00	1.10	2.24	2.00
5	1.97	0.27	2.20	1.86

Endogenous Financial Constraint and Investment-Cash-Flow Sensitivity

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Endogenous Financial Constraint and Investment-Cash-Flow Sensitivity

Abstract

This paper studies a dynamic investment model with moral hazard. The moral hazard problem implies an endogenous financial constraint on investment that makes the firm's investment sensitive to cash flows. I show that the production technology and the severity of the moral hazard problem substantially affect the dependence of the investment-cash-flow sensitivity on the financial constraint. Specifically, if the production technology exhibits almost constant returns to scale in capital or the moral hazard problem is relatively severe, the dependence is *negative*. Otherwise, the pattern is reversed to some extent. Moreover, the calibrated benchmark model can quantitatively account for the negative dependence of investment and Tobin's Q on size and age observed in the data.

Keywords: Dynamic moral hazard, financial constraint, investment-cash-flow sensitivity.

1 Introduction

The literature documents a significant correlation between firms' investments and their cash flow after controlling for Tobin's Q, which contradicts the traditional Q theory in a frictionless economy. People attribute this investment-cash-flow sensitivity to the firm's financial constraint on investment implied by various frictions. The subject of research over the past several decades studies the dependency of investment cash-flow sensitivity on the tightness of the financial constraint. Understanding this dependence enables us to see whether this sensitivity is a quantitative measure of the financial constraint, which substantially affects firm behavior but is difficult to observe. However, researchers obtains mixed results about this dependence. For example, Fazzari, Hubbard, and Petersen (1988) and Gilchrist and Himmelberg (1995) find that more financially constrained firms exhibit greater investment-cash-flow sensitivities,[1] whereas Kaplan and Zingales (1997) and Cleary (1999) find the opposite pattern. Until now, this research question has remained open.[2] Surprisingly, no attempt has been made to understand whether a unique correlation exists between investment-cash-flow sensitivity and the financial constraint in general or to determine what factors make this correlation positive or negative. Some of these important factors could potentially vary across industries, countries, or time.

In this paper, I introduce moral hazard into an otherwise standard firm investment model where investment is sensitive to cash flows because of an endogenous financial constraint on investment. Under this theoretical framework, I quantitatively show that whether the magnitude of the investment-cash-flow sensitivity increases or decreases with the tightness of the financial constraint depends on two important factors: (1) the returns to scale in capital of the production technology and (2) the severity of the moral hazard problem. Specifically, if the returns to scale in capital are close to constant or the moral hazard problem is relatively severe, the sensitivity *decreases* with the financial constraint; otherwise, the dependence is reversed to some extent.

In my model, the economy consists of a large number of entrepreneurs, each of whom is endowed with a technology that allows him to produce consumption goods from capital over a long time horizon. The cash flows generated by the firms are subject to random shocks, and the firms incur temporary losses. Since the entrepreneurs do not have initial wealth, they ask an investor for financing to cover their losses so that their firms can maintain capital input. Because the cash flows shocks are not observable to the investors, so that the entrepreneurs could misreport the temporary losses and divert cash flows, creating moral

[1]Based on this result, some researchers use investment-cash-flow sensitivity as an indicator of the financial constraint (e.g. Hoshi, Kashyap, and Scharfstein (1991) and Almeida and Campello (2007)).

[2]Kadapakkam, Kumar, and Riddick (1998) and Vogt (1994) find that large firms that seem to be less financially constrained exhibit greater investment-cash-flow sensitivity.

hazard. Therefore, to deter hidden diversions, an entrepreneur's stake in the firm, the fraction of the firm's future cash flows that belongs to him, is sensitive to the reported cash flows. The entrepreneur is protected by limited liability so that the firm has to be liquidated when this stake reaches zero after a sequence of negative cash flow shocks. Since liquidation is inefficient, an agency cost of incentive provisions arises, which implies a financial constraint on capital input. This is because a higher level of capital input increases the cash flows overseen by the entrepreneur and thus requires more intense incentive provisions, which raise the liquidation probability. As a result of the pay-performance sensitivity, positive cash flow shocks raise the entrepreneur's stake in the firm, relax the financial constraint, and allow the firm to input a higher and more efficient level of capital. However, negative shocks lower this stake, tighten the financial constraint, and force the firm to cut capital input. Consequently, the model endogenously generates an investment-cash-flow sensitivity.

If the production technology exhibits almost constant returns to scale, the marginal product of capital does not diminish at high capital levels. Therefore, when the financial constraint is relaxed upon a positive cash-flow shock, capital input increases significantly. In addition, a high level of capital requires more intense incentive provisions so that the financial constraint is more responsive to cash flow shocks. Consequently, less constrained firms, which deploy more capital, exhibit larger investment-cash-flow sensitivities. Clearly, if the marginal product of capital diminishes significantly at high levels of capital, in a less constrained firm, the investment would respond less to cash flow shocks. I measure the severity of the moral hazard problem by the volatility of the unobservable cash flow shocks, the noise in the entrepreneur's cash flow reports. If the noise level is high, a less constrained firm is still subject to a significant liquidation probability so that cash flow shocks still have significant impacts on the financial constraint. Given the high levels of capital input and the incentive provisions, investment is more responsive to cash flow shocks in this firm. However, if the noise level is low, in a less constrained firm, cash flow shocks have a weaker influence on the financial constraint and on investment. My results suggest that it is important to control for the production technology and the severity of the moral hazard problem when studying how investment-cash-flow sensitivity depends on the financial constraint.

Under my framework, a young firm is subject to a tighter financial constraint. In the early stage of its life cycle, the firm's growth and investment are driven by the relaxation of the financial constraint and the progress of productivity. In the late stage, when the firm grows larger and older, the agency cost vanishes and the financial constraint is relaxed so that its investment and growth rely only on the progress of productivity. Therefore, large and old firms invest less, compared with their small and young counterparts. Since Tobin's Q is approximately the marginal product of capital, it decreases as the firm grows large. The negative dependence of investment and Tobin's Q on size and age are consistent with the data.

The main topic of this paper relates to whether investment-cash-flow sensitivity is a quantitative measure of the financial constraint. The theoretical framework builds on the rapidly growing literature on continuous-time dynamic contract design models (DeMarzo and Sannikov (2006), Biais, Mariotti, Rochet, and Villeneuve (2010), Williams (2011), DeMarzo, Fishman, He, and Wang (2012), and Zhu (2013)). In contrast to these studies, this paper studies the implications of the moral hazard problem on the pattern of investment-cash-flow sensitivity and the endogenous financial constraint on investment. This paper also relates to studies of the dependence of investment-cash-flow sensitivity on firm characteristics including Gomes (2001), Alti (2003), Moyen (2004), Lorenzoni and Walentin (2007), and Abel and Eberly (2011). None of these papers consider moral hazard as a microeconomic foundation of the financial constraint. Ai, Li, and Li (2017) also study the financial constraint and the implied investment-cash-flow sensitivity based on a dynamic moral hazard model. They focus on the failure of traditional Q theory and the negative dependence of the investment-cash-flow sensitivity on firm size and age. In contrast, this paper observes the factors of the production process that affect the relation between the investment-cash-flow sensitivity and the financial constraint.

In terms of modeling the financial constraint, this paper relates to Clementi and Hopenhayn (2006). Both papers study a financial constraint implied by a dynamic moral hazard problem which restricts the scale of the production financed by the outside investor.[3] However, they pay attention to the properties of firm dynamics implied by the financial con-

[3]Albuquerque and Hopenhayn (2004) model the financial constraint in a similar way, but their constraint is a result of the limited commitment in financial contracts.

straint. This paper studies how the investment-cash-flow sensitivity depends on the financial constraint. Furthermore, this paper emphasizes the quantitative implications of the model and shows how the calibrated benchmark model matches the data.

The rest of the paper is organized as follows. In Section 2, I lay out the model, and in Section 3, I characterize the optimal contract. In Section 4, I show how investment-cash-flow sensitivity depends on the endogenous financial constraint; Section 5 interprets the investment-cash-flow sensitivity regressions based on the simulation data. Section 6 shows how I calibrate the benchmark model and how it fits the data, and Section 7 concludes.

2 The model

The model is an extension of DeMarzo and Sannikov (2006). The time horizon of the model is $[0, \infty)$. A unit measure of risk-neutral entrepreneurs arrives at the economy per unit of time. Each entrepreneur is endowed with a technology that allows him to produce consumption goods from capital over a long time horizon by establishing a firm. Let us consider a firm established at time zero. The productivity of the firm at $t \geq 0$ is

$$Z_t = \exp(\mu t),$$

with μ being the productivity growth rate. As in DeMarzo and Sannikov (2006), let Y_t be the quantity of cash flows that the firm generates up to time t. Given a sequence of capital inputs, $\{K_t\}$, the cash flow rate at time t is given by

$$dY_t = Z_t^{1-\alpha} K_t^{\alpha} + K_t \sigma dB_t. \tag{1}$$

Here, $\alpha \in (0, 1)$ is the capital share of the production technology, $\{B_t\}$ is a standard Brownian motion characterizing the idiosyncratic cash flow shocks, and $\sigma > 0$ is the rate of volatility. Each unit of capital inputs requires a user's cost, $r + \delta + \kappa$, per unit of time, where $r > 0$, $\delta > 0$, and $\kappa > 0$ are the interest rate, the capital depreciation rate, and the death rate of existing entrepreneurs, respectively. In this model, I assume that entrepreneurs are hit by death shocks independently with a fixed Poisson rate $\kappa > 0$. Upon the death shock, the entrepreneur and the firm exit the economy. I denote the Poisson time of the death shock by τ.[4]

[4]Because of the birth and death of the firms, the economy has a steady state.

The entrepreneur does not have initial wealth and asks an investor to finance the cost of capital. The investor provides financing by offering a lending contract, $(\{C_t\}, \{K_t\}, T)$. Specifically, C_t is the total transfers that the investor pays to the entrepreneur up to time t, and dC_t is the instantaneous rate of transfer, which has to be non-negative because the entrepreneur is protected by limited liability. The term K_t is the quantity of capital financed by the investor at t, and T is the time to liquidate the firm. All three terms depend on the entire history. Under the contract, the entrepreneur reports and hands over the firm's cash flows to the investor. Upon liquidation, the firm does not generate residual values.[5]

Moral hazard arises because the cash flow shocks are not observable to the investor. Therefore, the entrepreneur could misreport and secretly divert cash flows to increase consumption. Under the contract, if the entrepreneur diverts D_t for $t \geq 0$, his expected utility is

$$E_0 \left[\int_0^{\tau \wedge T} e^{-\beta t} \left(D_t dt + dC_t \right) \right],$$

where $\beta > 0$ is his discount rate, and E_0 is the time zero expectation operator.[6] On the other hand, the investor's expected payoff is

$$E_0 \left[\int_0^{\tau \wedge T} e^{-rt} \left[\left(Z_t^{1-\alpha} K_t^{\alpha} - (r + \delta + \kappa) K_t - D_t \right) dt - dC_t \right] \right]. \tag{2}$$

I assume that $\beta > r$ so that the investor is more patient. Since there is no upper bound on the amount of the cash flows that the entrepreneur could divert, I focus on the incentive-compatible contract under which he is induced to truthfully report the cash flows.[7] The investor designs the optimal incentive-compatible lending contract to maximize her expected payoff, (2), and promises the entrepreneur an initial expected utility U_0. To guarantee that the firm value to the investor is finite, I make the following assumption.

Assumption 1. $r + \kappa > \mu$.

Clearly, the larger the parameter σ, the harder it is to infer the actual cash flows from the entrepreneur's reports. Therefore, σ indicates the severity of the moral hazard problem. Moreover, given σ, the volatility is proportional to the capital stock K_t. Intuitively, when

[5] This assumption is not essential to the key results of the paper.

[6] Obviously, the probability basis of this expectation depends on the entrepreneur's diversion behavior.

[7] See DeMarzo and Sannikov (2006) for the argument about the optimality of doing so.

the firm is larger and the entrepreneur oversees a greater quantity of cash flows, more intense incentive provisions are required to deter hidden diversions. Since the model allows long-run growth of the firms, this assumption prevents firms from growing out of the moral hazard problem.

3 The optimal contract

3.1 Normalization and incentive compatibility

By following the literature, I define the entrepreneur's continuation utility

$$U_t = E_t \left[\int_t^{\tau \wedge T} e^{-\beta(s-t)} dC_s \right] \text{ for } t \in [0, \tau \wedge T]$$

as one of the state variables. Let $V(Z, U)$ be the value function of the investor's maximization problem. Given the homogeneity of this problem, the value function satisfies

$$V(Z, U) = Zv\left(\frac{U}{Z}\right),$$

where $v(u)$ is the normalized value function and $u = \frac{U}{Z}$ is the entrepreneur's normalized continuation utility, which is the ratio of his future payments to the scale of production and can be interpreted as the entrepreneur's stake in the firm. Accordingly, I define the normalized capital input and transfer payment to the entrepreneur, $k_t = \frac{K_t}{Z_t}$ and $dc_t = \frac{dC_t}{Z_t}$, respectively. The Martingale representation theorem implies the following law of motion of u_t.

Lemma 1. *Suppose the contract $(\{C_t\}, \{K_t\}, T)$ is incentive compatible and the entrepreneur's normalized continuation utility satisfies*

$$du_t = u_t (\beta + \kappa - \mu) dt - dc_t + g_t \sigma dB_t. \tag{3}$$

Here $\{g_t\}$ is a predictable process such that $\{g_t Z_t\}$ is square integrable.

Proof. See Appendix A. □

Since dB_t represents the noise in the entrepreneur's cash-flow reports, g_t is the sensitivity of his continuation utility with respect to his reports. Therefore, g_t determines the entrepreneur's incentives to truthfully report cash flows or not report them. Hence, we have the following incentive compatibility condition.

Lemma 2. *A contract* $(\{C_t\}, \{K_t\}, T)$ *is incentive compatible if and only if*

$$g_t \geq k_t \text{ for all } t \in [0, \tau \wedge T]. \tag{4}$$

Proof. See Appendix B. □

According to (4), g_t has to be no lower than the normalized working capital level, k_t. Intuitively, a greater pay-performance sensitivity is required to deter hidden diversions when the entrepreneur oversees a larger scale of production. In fact, this condition is binding under the optimal contract because the unnecessary exposure of u_t to the cash flow shocks reduces efficiency, as I prove in Proposition 2 in Appendix C.

3.2 A brief characterization of the optimal contract

In this subsection, I briefly describe the optimal contract and the normalized value function, $v(u)$, and leave the detailed discussions to Appendix C. There is an upper bound $\hat{u}$, such that, under the optimal contract, if $u_t \in [0, \hat{u}]$, the investor does not pay the entrepreneur and u_t evolves according to

$$du_t = u_t (\beta + \kappa - \mu) dt + k_t \sigma dB_t. \tag{5}$$

Once u_t reaches zero, the firm is liquidated, namely, $T = \inf \{t : u_t = 0\}$. Given any history, if $u_t \geq \hat{u}$, a lump-sum transfer, $dc_t = u_t - \hat{u}$, is paid to the entrepreneur immediately so that the transfers that need to be paid in the future decrease and u_t reflects back to $\hat{u}$. Over $[0, \hat{u}]$, $v(u)$ is strictly concave and satisfies

$$0 = \max_{k \geq 0} k^{\alpha} - (r + \delta + \kappa) k - (r + \kappa - \mu) v(u) + (\beta + \kappa - \mu) uv'(u) + \frac{1}{2} v''(u) \sigma^2 k^2. \tag{6}$$

Over $[\hat{u}, \infty)$, $v'(u) = -1$. The optimal capital input $k(u)$ maximizes the objective function on the right-hand side of (6).

3.3 The agency cost and endogenous financial constraint

In this subsection, I show how the moral hazard problem implies a financial constraint on capital financing. Recall that the entrepreneur's stake, u, is the expected present value of the payments that the entrepreneur is going to receive under the lending contract. Hence, to deter hidden diversions, this stake has to be sensitive to the cash flows reported by him so that it increases upon positive shocks and decreases upon negative ones. Since the entrepreneur is protected by limited liability, as in DeMarzo and Sannikov (2006),[8] liquidation has to be used as ultimate punishment when u decreases to zero after a sequence of negative cash flow shocks,[9] even though liquidation is inefficient due to the forgone future income. When deciding the level of capital to finance, the investor needs to take into account the possibility of liquidation. According to Lemma 2, when the firm is producing on a larger scale, stronger incentive provisions are needed to deter diversions, which increase the volatility of u. Hence, a higher level of capital would raise the probability of liquidation, and then an agency cost of financing capital arises, which endogenously implies a financial constraint. This financial constraint forces the investor to choose a capital level strictly lower than the first-best one.[10] Obviously, as the level of u increases, the risk of liquidation diminishes, and the agency cost decreases, allowing the investor to finance a higher and more efficient level of capital.

Technically, if there were no moral hazard, the investor would always choose the first-best level of k that maximizes the operating profit,

$$k^{\alpha} - (r + \delta + \kappa)\, k.$$

With moral hazard, according to (6) it maximizes

$$k^{\alpha} - (r + \delta + \kappa)\, k + \frac{1}{2} v''(u) \sigma^2 k^2.$$

The last term in the expression above stands for the agency cost of incentive provisions. Since $v(u)$ is strictly concave, the larger the $|v''(u)|$, the more k is restricted from its efficient level, and the tighter the financial constraint.[11]

[8]See He (2009) and DeMarzo, Fishman, He, and Wang (2012) for the argument with a geometric Brownian motion cash-flow process.

[9]Due to the limited liability of the entrepreneur, negative transfer payments are not feasible. Therefore, the only way to creditably make his stake zero is to liquidate the firm and terminate the contract, because, otherwise, the entrepreneur can always divert cash flows as long as the firm is being financed with capital and producing.

[10]We discuss the first-best case in Appendix D.

[11]In fact, the concavity of $v(u)$ is the “implied risk aversion” in my model, because the volatility of u_t raises the probability of the liquidation and reduces efficiency.

I plot the $|v''(u)|$ and the normalized capital input function, $k(u)$, in the upper and lower panels of Figure 1, respectively. As seen in the diagram, $|v''(u)|$ monotonically decreases with u over $[0, \hat{u}]$. Obviously, the probability of liquidation and the agency cost diminish as u increases. Thus, the capital input increases with u as the financial constraint is relaxed and reaches its first-best level, k^*, at $\hat{u}$ where the agency cost vanishes. Therefore, to alleviate the agency cost and relax the financial constraint, the transfer payments promised to the entrepreneur are deferred even though he is less patient, so that u_t drift up with rate $\beta + \kappa - \mu > 0$,[12] and the investor can choose a higher and more efficient capital level to finance.

I assume that the investor has full bargaining power when offering the contract. Therefore, she chooses the initial continuation utility U_0 to maximize the firm value to her; namely, the initial normalized continuation utility, u_0, satisfies $u_0 = \arg\max_{u \in [0,\hat{u}]} v(u)$.

3.4 The investment-cash-flow sensitivity

In this subsection, I derive a simple expression for the investment-cash-flow sensitivity under the optimal contract. Instantaneously, the investment-to-capital ratio is $\frac{dK_t}{K_t} + \delta dt$. Given (5), Itô's lemma implies that, for all $t \in [0, \tau \wedge T]$,

$$\frac{dK_t}{K_t} = \left[\mu - \delta + \frac{k'(u_t)(\beta + \kappa - \mu) u_t + \frac{1}{2} k''(u_t) \sigma^2 k(u_t)^2}{k(u_t)}\right] dt + k'(u_t) \sigma dB_t. \qquad (7)$$

On the other hand, (1) implies

$$\frac{dY_t}{K_t} = k(u_t)^{\alpha - 1} dt + \sigma dB_t. \qquad (8)$$

Hence, (7) and (8) straightforwardly imply the following result.

Proposition 1. *Under the optimal contract, the instantaneous investment-cash-flow sensitivity is*

$$\frac{Cov\left(\frac{dK_t}{K_t}, \frac{dY_t}{K_t}\right)}{Var\left(\frac{dY_t}{dK_t}\right)} = k'(u_t). \qquad (9)$$

If there were no moral hazard, the normalized capital input would be time invariant so that $\frac{dK_t}{K_t} = dZ_t = \mu dt$. Therefore, we have the following corollary.

[12]See the law of motion of u_t under the optimal contract, (5).

Corollary 1. *In the first-best case, the investment-cash-flow sensitivity is zero.*

With moral hazard, positive cash flow shocks relax the constraint, enabling the investor to finance more capital, whereas negative shocks tighten the constraint, forcing the investor to reduce capital. Consequently, the model endogenously implies a positive investment-cash-flow sensitivity.

4 Relationship between investment-cash-flow sensitivity and financial constraint

Given the expression for the investment-cash-flow sensitivity in Proposition 1, I study how the returns to scale in capital of the production technology, α, and the severity of the moral hazard problem, σ, affect the relationship between the investment-cash-flow sensitivity and the financial constraint. Notice that the financial constraint directly restricts the firm's capital financing. The tighter the financial constraint, the further the normalized capital level, $k(u)$, is below the first-best one, k^*. Therefore, the deviation of $k(u)$ from k^*, $(k^* - k(u))/k^*$, is a straightforward measure of the tightness of the constraint, which I use in the interpretation. Specifically, based on the calibrated benchmark model introduced in Section 6, I take three different levels of α and σ, calculate the optimal contract for each, and plot $k'(u)$ as a function of the deviation of capital financing in the left and right panels of Figure 2, respectively.

As seen in the left panel, when the capital share, α, is close to one, the dependence of the investment-cash-flow sensitivity on the financial constraint significantly decreases. As α decreases, the significance of this pattern diminishes, and the dependence increases in some region if α is too small. To understand this phenomenon, notice that, in a less constrained firm, a positive cash-flow shock, dB_t, induces a larger increase in u_t and a greater relaxation of the financial constraint because of the positive dependence of pay-performance sensitivity on the capital level (incentive constraint (4)). If the production technology exhibits almost constant returns to scale, the marginal product of capital does not diminish significantly; thus, a greater relaxation of the financial constraint implies a larger increase in capital. Therefore, a less constrained firm exhibits a greater investment-cash-flow sensitivity. It is easy to see that if the marginal product of capital diminishes too quickly when the firm deploys more capital, the pattern can be the opposite, as shown in the diagram.

According to the right panel of Figure 2, if the contract is subject to a severe moral hazard problem, the magnitude of the investment-cash-flow sensitivity decreases with the tightness of the financial constraint. If the severity decreases, the pattern is reversed in some region. The tightness of the financial constraint is determined by the probability of the liquidation. According to the law of motion of u_t (equation (5)), the liquidation probability over a unit length of time is approximately the left tail, cut off at zero, of a normal distribution with mean u_t and standard deviation $\sigma k(u_t)$. Given the bell shape of the normal distribution, a large σ implies that the liquidation probability is still sensitive to the changes in u_t even if u_t is at a high level (less constrained). Moreover, the incentive provision is multiplicative in $k(u_t)$. Therefore a unit size of positive cash flow shock induces a larger increase in u_t and $k(u_t)$. As a result, the investment-cash-flow sensitivity increases with u_t and decreases with the tightness of the financial constraint. However, if σ is relatively small, the liquidation probability is sensitive to cash flow shocks only if u_t is close to zero (severely constrained). Hence, as u_t goes up, it quickly becomes insensitive to the cash flow shocks, and the investment-cash-flow sensitivity diminishes, as indicated by the solid curve in the right panel of Figure 2.

5 Investment-cash-flow sensitivity regressions

By using the simulation data, I regress the investment-to-capital ratio on Tobin's Q and the cash-flow-to-capital ratio to study how the coefficient on the cash-flow-to-capital ratio depends on the following three firm characteristics: (1) the tightness of the financial constraint, (2) the firm size, and (3) the firm age. I particularly want to see how these dependence are affected by the parameters, α and σ. Let the integer $\bar{t} = 0, 1, 2, ...,$ be the index of year, and let $K_{\bar{t}}$ and $Q_{\bar{t}}$ be the capital input level and Tobin's Q at the beginning of year $\bar{t}$, respectively. The investment-to-capital ratio over this year is

$$i_{\bar{t}} = \ln(K_{\bar{t}+1}) - \ln(K_{\bar{t}}) + \delta,$$

and the cash-flow-to-capital ratio is $cf_{\bar{t}} = \frac{CF_{\bar{t}}}{K_{\bar{t}}}$ with

$$CF_{\bar{t}} = \int_{\bar{t}}^{\bar{t}+1} dY_t = \int_{\bar{t}}^{\bar{t}+1} \left[K_t^{\alpha} Z_t^{1-\alpha} dt + \sigma K_t dB_t\right].$$

As in Section 4, based on the benchmark parameter values calibrated in Section 6, I choose three different values of α and σ, respectively. For each specification of the parameter values, I simulate the optimal contract,[13] and I divide firm samples into five groups with an equal number of samples in each group according to each of the three firm characteristics listed above. I run the investment-cash-flow sensitivity regression in each group. As in the literature, the repression equation is[14]

$$i_{\bar{t}} = \beta_0 + \beta_Q Q_{\bar{t}} + \beta_{CF} cf_{\bar{t}} + \varepsilon.$$

In Table 1, for different values of α and σ, I report the regression coefficients on the cash-flow rate, β_{CF}, in different groups subject to different levels of the financial constraint.[15] As in Section 4, the tightness of the financial constraint is measured by the deviation of capital financing from its first-best level.

As seen in Table 2, when α or σ is small, less constrained groups generally exhibit smaller coefficients. As α or σ goes up, the pattern becomes the opposite. Under my framework, firm size and age can be proxies for the financial constraint. Intuitively, when a new firm is established, it is subject to a tight financial constraint because the entrepreneur's stake in the firm is small and the capital level is low. As the firm grows larger and older, the entrepreneur's stake drifts up (equation (5)) and the financial constraint is relaxed. As a result, firm size and age decrease with the tightness of the financial constraint so that large and old firms are less constrained.

In Tables 2 and 3, firm samples are divided into different groups according to their size and age, respectively. As expected, if the parameter, α or σ, is small, the regression coefficient on the cash flow rate significantly decreases with size and age. The significance diminishes as the parameter value increases, and the pattern is reversed if the parameter value is too

[13] I simulate an economy consisting of $20,000$ firm positions for 350 years and collect the firm-year samples in the last 50 years making sure that the cross-sectional distribution of the firms over the state space is time invariant. For each simulation, I obtained approximately $900,000$ samples.

[14] In the model, there is no firm or year fixed effect. Notice that the coefficients are significantly larger than that documented in the empirical literature. There could be some additional random shocks in cash flows in reality that are not modeled in my framework.

[15] All the regression coefficients for the simulation data are significant, so I only report their values in the tables.

large.

6 Calibration of the benchmark model

I show how I calibrate the benchmark parameter values of the model and that the benchmark model quantitatively accounts for the negative dependences of the investment-to-capital ratio and Tobin's Q on firm size and age.

To calibrate the benchmark model, I first choose the interest rate $r = 4\%$ to be the average return of risky and risk-free assets in the U.S. postwar period, and the capital depreciation rate $\delta = 10\%$, the depreciation rate documented in the real business cycle literature.[16] The rest of the parameters are chosen by matching the moments estimated from the manufacturing firms in the Compustat data set for the period 1967 to 2015, which Table 4 summarizes.

Specifically, I choose the parameter of the returns to scale of capital $\alpha = 0.8$ to match the median cash-flow-to-capital ratio in the data, which is 35%. The death rate $\kappa = 0.05$ per annum is the average death rate of the firms in the data. I set the productivity growth rate $\mu = 0.049$ to match the median growth rate of the old firms,[17] which is 7.6% and $\beta = 0.08$ to match the median growth rate of the entire data set, which is 8.8%. I choose the volatility of the cash flow shocks, $\sigma = 0.35$, to match the standard deviation of the cash-flow-to-capital ratio in the data, which is 44.2%.

For both the Compustat data and the simulation data, I divide firm-year samples into five groups according to their initial sizes and ages in a year, respectively, with an equal number of samples in each group. Then, I evaluate the median investment-to-capital ratio in each group and report them in Table 5. The firms in the groups 1 to 5 are from small to large or from young to old. As seen in the table, the investment-to-capital ratio decreases with size and age and that pattern is consistent with the data. In fact, the negative dependence of the investment-to-capital ratio on size and age has been documented in empirical studies; for example, see Evans (1987) and Hall (1987). The economic mechanism behind this pattern in the model is clear. In early stage of a firm's life cycle, u_t is at a low level and the firm is subject to a tight financial constraint, which restricts the firm's investment. In this

[16] For example, see Mehra and Prescott (1985).

[17] I define old firms to be the those that are older than the median age.

early stage, the firm's growth and investment are driven by the relaxation of the financial constraint and the progress of productivity. Once the agency cost vanishes and the financial constraint is relaxed, the firm matures, and its growth and investment are driven only by productivity growth. Consequently, large and old firms grow slower than their small and young counterparts.

Consistent with the data, Tobin's Q decreases with size and age. Intuitively, in my model, Tobin's Q is largely determined by the marginal product of capital. Large and old firms are less constrained so that they deploy higher levels of capital and thus have lower marginal products of capital.

7 Conclusion

I propose a theoretical framework under which a moral hazard problem endogenously induces a financial constraint on investment so that investment responds to cash flow shocks. I show that, under this framework, whether the investment-cash-flow sensitivity increases or decreases with the tightness of the financial constraint substantially depends on the returns to scale in capital of the production technology and the severity of the moral hazard problem. If the technology exhibits almost constant returns to scale or the production is subject to a severe moral hazard problem, investment-cash-flow sensitivity is negatively correlated with the financial constraint; otherwise, the correlation could be positive. This result emphasizes the controls for the production technology and the monitoring structure of firm organization in future empirical studies. Moreover, the calibrated benchmark model can replicate the negative dependence of the investment-to-capital ratio and Tobin's Q on firm size and age that is observed in the data.

Appendix

A Proof of Lemma 1

Let Υ_t be the time-t conditional expectation of the entrepreneur's total utility under the contract. Then I have

$$\Upsilon = E_t\left[\int_0^T e^{-(\beta+\kappa)t}dC_t\right] = \int_0^t e^{-(\beta+\kappa)s}dC_s + e^{-(\beta+\kappa)t}U_t. \tag{A.10}$$

So $\{\Upsilon_t\}$ is an adapted martingale and the Martingale representation theorem implies

$$d\Upsilon_t = e^{-(\beta+\kappa)t}G_t\sigma dB_t \tag{A.11}$$

with $\{G_t\}$ being a predictable and square integrable process. Equation (A.10) and (A.11) implies the law of motion U_t and then (3) according to Itô's lemma if I define $g_t = \frac{G_t}{Z_t}$.

B Proof of Lemma 2

I show the following incentive compatibility condition which is equivalent to (4).

$$G_t \geq K_t \text{ for all } t \in [0, \tau \wedge T]. \tag{A.12}$$

Given any diversion policy of the entrepreneur, $\{D_t\}$, define $\left\{B_t^D\right\}$ such that

$$dB_t^D = \frac{dY_t - \left(Z^{1-\alpha}K_t^\alpha - D_t\right)dt}{\sigma K_t} \text{ for all } t \in [0, \tau \wedge T]. \tag{A.13}$$

Given any $t \in [0, \tau \wedge T]$, I define

$$\mathcal{G}_t^D = \int_0^t e^{-(\beta+\kappa)s}\left(D_s ds + dC_s\right) + e^{-(\beta+\kappa)t}U_t.$$

Intuitively, $\mathcal{G}_t$ is the time-t conditional expected utility of the manager if he diverts cash flows according to $\{D_t\}$ until t and then stops doing so. Clearly, $\mathcal{G}_0^D = U_0$. Condition (A.13) implies

$$e^{-(\beta+\kappa)t}d\mathcal{G}_t^D = D_t\left(1 - \frac{G_t}{K_t}\right)dt + G_t\sigma dB_t^D.$$

The second term on the right hand side is a martingale under the diversion policy $\{D_t\}$. The entrepreneur is not willing to divert cash flows if and only if $\left\{\mathcal{G}_t^D\right\}$ is a super martingale, and if and only if (A.12) is satisfied.

C Characterization of the optimal contract

I discuss the normalized value function, $v(u)$, and the optimal contract in more details which is extended from DeMarzo and Sannikov (2006) and He (2009). The law of motion of u_t, (3), implies that $v(u)$ satisfies the following HJB differential equation

$$0 = \max_{k,g\geq k,dc} \quad k^{\alpha} - (r+\delta+\kappa)\,k - (r+\kappa-\mu)\,v\,(u) + (\beta+\kappa-\mu)\,uv'\,(u) \\ +\frac{1}{2}v''\,(u)\,g^2\sigma^2 - (1+v'(u))\,dc \tag{A.14}$$

The investor can always lower the entrepreneur's normalized continuation utility instantaneously by delivering him a lump-sum transfer. Therefore,

$$v(u) \geq v(u-dc) - dc \text{ for all } dc > 0 \text{ and } v'(u) \geq 1.$$

Proposition 2. *The optimal contract promising the manager initial continuation utility* $U_0 = u_0 Z_0 = u_0$ *takes the following form. For* $t \leq \tau \wedge T$, *when* $u_t \in (0, \hat{u})$, u_t *evolves according to* (5), $dc_t = 0$, *and* $k_t = k\,(u_t)$, *which is the maximizer of the right-hand side of* (6) *with level* u_t; *when* $u_t > \hat{u}$, $dc_t = u_t - \hat{u}$, *reflecting* u_t *back to* $\hat{u}$. *The firm is liquidated at time* T, *the time when* u_t *reaches zero. The normalized value function,* v, *satisfies* (6) *over* $[0, \hat{u}]$ *with boundary conditions,* $v(0) = 0$, $v'\,(\hat{u}) = -1$, $v''\,(\hat{u}) = 0$, *and*

$$v\,(\hat{u}) = \frac{\pi^*}{r+\kappa-\mu} - \frac{\beta+\kappa-\mu}{r+\kappa-\mu}\hat{u} \text{ with } \pi^* = (1-\alpha)\left(\frac{\alpha}{r+\delta+\kappa}\right)^{\frac{\alpha}{1-\alpha}}. \tag{A.15}$$

For $u \geq \hat{u}$, $v(u) = v\,(\hat{u}) - (u - \hat{u})$. *Moreover,* v *is strictly concave over* $[0, \hat{u}]$.

The proof is divided into the following three lemmas. In the first lemma I show the concavity of the normalized value function.

Lemma 3. *The normalized value function* v *satisfying HJB* (6), *the boundary conditions* $v'\,(\hat{u}) = -1$, (A.15), *and* $v''\,(\hat{u}) = 0$ *is concave over* $[0, \hat{u}]$.

Proof. I divide both hand sides of (6) with respect to u based on the Envelope theorem and obtain

$$(\beta - r)\,v'(u) + (\beta+\kappa-\mu)\,uv''(u) + \frac{1}{2}v'''k(u)^2\sigma^2 = 0. \tag{A.16}$$

which implies, given the boundary conditions at $\hat{u}$,

$$v'''\left(\hat{u}\right)=\frac{2\left(\beta-r\right)}{k\left(\hat{u}\right)^{2}\sigma^{2}}>0.$$

So for any real number $\epsilon>0$ which is small enough, $v''\left(\hat{u}-\epsilon\right)<0$. Suppose that $v(u)$ is not concave and $\tilde{u}$ is the largest real number over $[0,\hat{u}]$ such that $v''\left(\tilde{u}\right)=0$. Then (6) implies that $k\left(\tilde{u}\right)=k^{*}$ and

$$v\left(\tilde{u}\right)=\frac{\pi^{*}}{\beta+\kappa-\mu}+\frac{\beta+\kappa-\mu}{r+\kappa-\mu}\tilde{u}v'\left(\tilde{u}\right). \tag{A.17}$$

Since $v'\left(u\right)>-1$ for all $u\in\left(\tilde{u},\hat{u}\right)$, (A.17) contradicts (A.15) and I have the desired result. □

Verification of $v(u)$ and the described contract are divided into Lemmas 4 and 5. Lemma 4 shows that the payoffs indicated by $v(u)$ is achievable under the described contract.

Lemma 4. *Under the contract described in Proposition 2, the normalized firm value is characterized by $v(u)$.*

Proof. Clearly, under the described contract, u_t follows (5). Now suppose that $U_0=u_0\in[0,\hat{u})$. I define

$$\Psi_{t}=\int_{0}^{t}e^{-(r+\kappa)s}\left[Z_{s}^{\alpha}K_{s}^{1-\alpha}-\left(r+\delta+\kappa\right)K_{s}-dC_{s}\right]+e^{-(r+\kappa)t}Z_{t}v\left(u_{t}\right)\text{ for any }t\in[0,\tau\wedge T]$$

and, obviously, $\Psi_0=Z_0v\left(u_0\right)=v\left(u_0\right)$. Then, according to Itô's lemma,

$$e^{-(r+\kappa)t}d\Psi_{t}=Z_{t}\left[\begin{array}{c}k_{t}^{\alpha}-\left(r+\delta+\kappa\right)k_{t}-\left(r+\kappa-\mu\right)v\left(u_{t}\right)+\left(\beta+\kappa-\mu\right)u_{t}v'\left(u_{t}\right)\\+\frac{1}{2}v''\left(u_{t}\right)k_{t}^{2}\sigma^{2}-\left(1+v'(u_{t})\right)dc_{t}+v'\left(u_{t}\right)k_{t}\sigma dB_{t}\end{array}\right]. \tag{A.18}$$

Under the contract, $dc_t\neq0$ if and only if $v'\left(u_t\right)=-1$ and k_t is the optimal solution of the maximization problem on the right hand side of (6). So (A.18) is simplified to

$$e^{-(r+\kappa)t}d\Psi_{t}=Z_{t}v'\left(u_{t}\right)k_{t}\sigma dB_{t},$$

and then $\{\Psi_t\}$ is an adapted martingale. Therefore, the expected payoff under the contract is $E_0\left[\Psi_\infty\right]=\Psi_0=v\left(u_0\right)$ and I have the desired result. □

Lemma 5. *Any incentive compatible contract promising the entrepreneur expected utility $U_0=Z_0u_0=u_0$ cannot generate an expected payoff larger than $Z_0v\left(u_0\right)=v\left(u_0\right)$.*

Proof. Because of the limited liability of the entrepreneur, the firm has to be liquidated when u_t hits zero. I denote the hitting time by T_0 which could be infinite. Let $\left(\left\{\hat{C}_t\right\}, \left\{\hat{K}_t\right\}, \hat{T}\right)$ be an alternative incentive compatible contract promising the entrepreneur initial expected utility u_0 and I use hatted letters denote corresponding terms under this contract. For any $t \in \left[0, \tau \wedge \hat{T} \wedge T_0\right]$, define

$$\hat{\Psi}_t = \int_0^t e^{-(r+\kappa)s} \left[\left(Z_s^\alpha \hat{K}_s^{1-\alpha} - (r+\delta+\kappa)\hat{K}_s\right) ds - d\hat{C}_s\right] + e^{-(r+\kappa)t} Z_t v(\hat{u}_t).$$

Obviously, $\hat{\Psi}_0 = Z_0 v(u_0) = v(u_0)$. Then

$$e^{-(r+\kappa)t} d\hat{\Psi}_t = Z_t \begin{bmatrix} \hat{k}_t^\alpha - (r+\delta+\kappa)\hat{k}_t - (r+\kappa-\mu) v(\hat{u}_t) + (\beta+\kappa-\mu)\hat{u}_t v'(\hat{u}_t) \\ +\frac{1}{2} v''(\hat{u}_t)\hat{g}_t^2 \sigma^2 - (1+v'(\hat{u}_t)) d\hat{c}_t + v'(\hat{u}_t)\hat{g}_t \sigma dB_t \end{bmatrix}.$$

Concavity of $v(u)$ (Lemma 3), the fact that $v' > -1$, and the HJB (6) imply that $\left\{\hat{\Psi}_t\right\}$ is a super martingale. Therefore the expected payoff of the investor under the alternative contract is

$$E_0\left[\hat{\Psi}_{\tau\wedge\hat{T}\wedge T_0}\right] \le E_0\left[\hat{\Psi}_0\right] = Z_0 v(u_0) = v(u_0).$$

□

D The first-best case

In this section, I consider the first-best case in which there is no moral hazard. Obviously, in this case, the firm is never liquidated and $K_t = k^* Z_t$ for $t \le \tau$ with

$$k^* = \arg\max_k k^\alpha - (r+\delta+\kappa) k = \left(\frac{\alpha}{r+\delta+\kappa}\right)^{\frac{1}{1-\alpha}},$$

and the instantaneous expected operating profit rate is $\pi^* Z_t$ with

$$\pi^* = (1-\alpha)\left(\frac{\alpha}{r+\delta+\kappa}\right)^{\frac{\alpha}{1-\alpha}}.$$

Therefore, the net present value of the total cash flows generated by the firm is $Z_0 \frac{\pi^*}{r+\kappa-\mu} = \frac{\pi^*}{r+\kappa-\mu}$. Since the investor is more patient, it is optimal to pay off all the transfers promised to the entrepreneur at $t = 0$. Therefore, the first-best normalized value function $v^{FB}(u)$ satisfies

$$v^{FB}(u) = \frac{\pi^*}{r+\kappa-\mu} \text{ for all } u \ge 0.$$

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In Table 6, I report the median Tobin's Q in each size or age group.

Figure 1.

Agency Cost and Optimal Capital Input.

In the upper panel I plot the absolute value of the second-order-derivative of v as a function of the entrepreneur's normalized continuation utility, u. In this model, $|v''(u)|$ reflects the agency cost. In the bottom panel I plot the normalized capital input, k, as a function of u. The agency cost diminishes at level $\hat{u}$ and the normalized capital input reaches its first-best level.

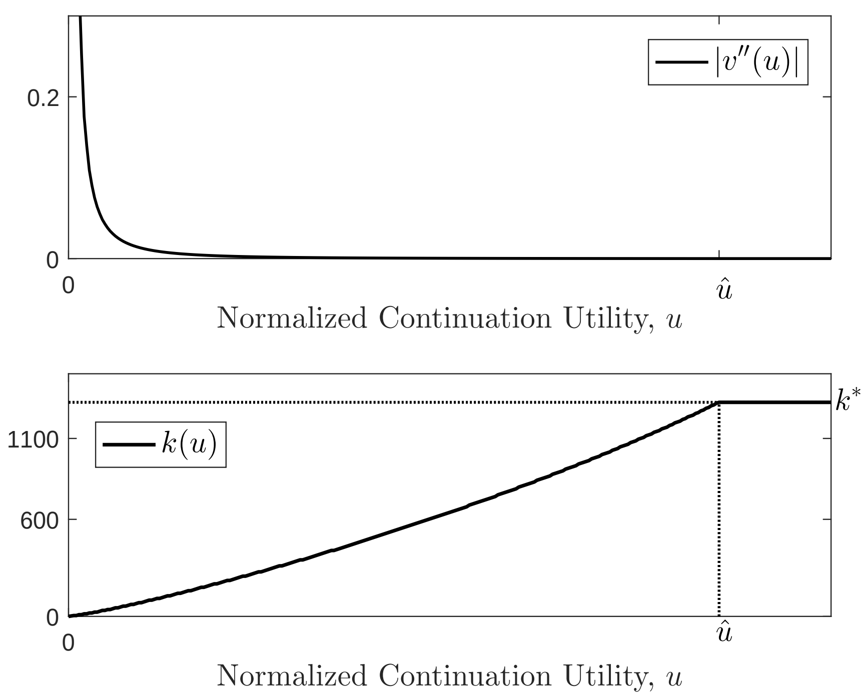

Figure 2.

Dependence of Investment-Cash-Flow Sensitivity on Tightness of Financial Constraint.

In both panels, I plot the instantaneous investment-cash-flow sensitivity, defined in equation (9), as a function of the deviation of capital financing, which indicates the tightness of the financial constraint. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α and σ, and plot the diagrams in the left and right panels respectively.

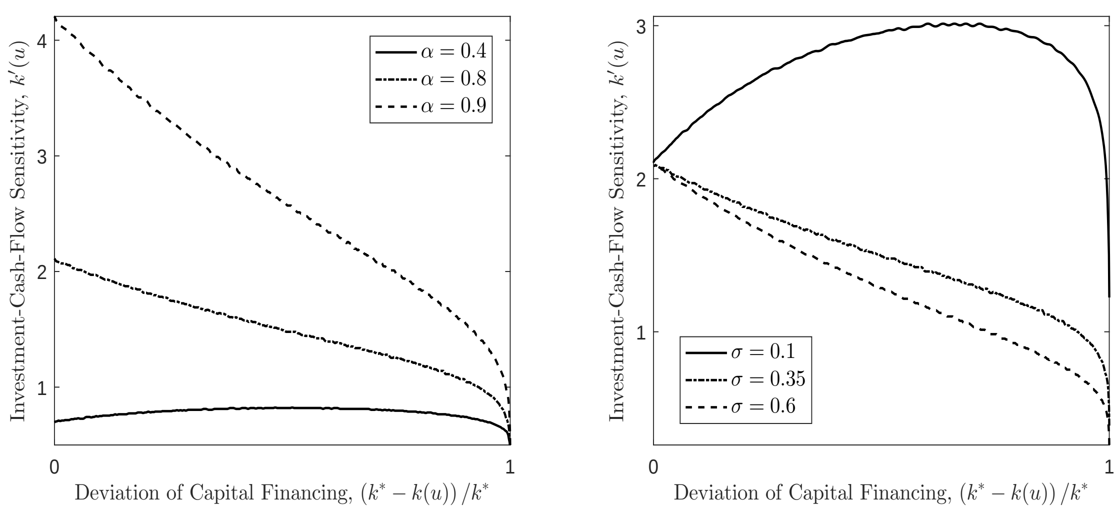

Table 1.

Dependence of investment-cash-flow sensitivity regression coefficient on tightness of the financial constraint with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the financial constraints the group members subject to relaxes. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Constrained	0.68	0.84	0.88	2.22	0.83	0.51
2	0.76	0.95	1.04	2.35	0.95	0.60
3	0.76	1.04	1.13	2.33	1.04	0.66
4	0.63	1.17	1.27	1.96	1.17	0.77
Unconstrained	0.36	1.05	1.54	1.17	1.05	0.87

Table 2.

Dependence of investment-cash-flow sensitivity regression coefficient on firm size with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the size of the group members increases. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Small	0.70	0.88	0.93	2.26	0.88	0.54
2	0.74	0.97	1.05	2.32	0.97	0.61
3	0.68	1.06	1.14	2.14	1.06	0.68
4	0.58	1.12	1.30	1.83	1.13	0.78
Large	0.57	1.04	1.46	1.77	1.04	0.81

Table 3.

Dependence of investment-cash-flow sensitivity regression coefficient on age with different values of α and σ.

Simulated firm samples are divided into five groups with equal number in each group. From group 1 to 5, the age of the group members increases. Based on the calibrated benchmark model introduced in Section 6, I take three different levels of α, and interpret the investment-cash-flow sensitivity regression coefficients in different groups from column 2 to 4. I also take three different levels of σ and interpret the coefficients in different groups from column 5 to 7.

Group No.	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 0.9$	$\sigma = 0.1$	$\sigma = 0.35$	$\sigma = 0.6$
Young	0.73	1.00	1.11	2.29	1.00	0.65
2	0.67	1.02	1.17	2.14	1.02	0.68
3	0.64	1.02	1.20	2.05	1.02	0.69
4	0.63	1.02	1.22	2.00	1.02	0.70
Old	0.62	1.02	1.23	1.99	1.02	0.71

Table 4.

Calibrated parameter value and targeted moments.

Model-specific parameters		Value	Targeted Moments
α	Returns to scale parameter	0.800	Median cash flow rate of Computat firms
κ	Average death rate	0.050	Average death rate of Compustat firms
μ	Productivity growth rate	0.049	Median growth of old Compustat firms
β	Discount rate	0.08	Median growth rate of Compustat firms
σ	Volatility of unobservable shock	0.35	Volatility of cash flow rate of Compustat firms

Table 5.

Investment-to-capital ratio by size and age.

Firm samples observed in the data and generated in the simulation are respectively divided into five groups with equal number of samples in each group according to their size (column 2 and 3) and age (column 4 and 5). From group 1 to 5, size or age increases. I calculate the average investment-to-capital ratio in each group. In both the data and the simulation, the investment-to-capital ratio decreases with size and age. The model simulation is based on on the calibrated benchmark model introduced in Section 6.

	Size Group		Age Group	
Group No.	Data	Model	Data	Model
1	0.21	0.20	0.21	0.19
2	0.20	0.19	0.20	0.18
3	0.19	0.19	0.19	0.17
4	0.18	0.17	0.19	0.16
5	0.18	0.11	0.18	0.15

Table 6.

Tobin's Q by size and age.

Firm samples observed in the data and generated in the simulation are respectively divided into five groups with equal number of samples in each group according to their size (column 2 and 3) and age (column 4 and 5). From group 1 to 5, size or age increases. I calculate the average Tobin's Q in each group. In both the data and the simulation, the Tobin's Q decreases with size and age. The model simulation is based on on the calibrated benchmark model introduced in Section 6.

	Size Group		Age Group	
Group No.	Data	Model	Data	Model
1	3.03	6.21	2.89	3.24
2	3.00	3.33	2.78	2.77
3	2.24	2.21	2.36	2.31
4	2.00	1.10	2.24	2.00
5	1.97	0.27	2.20	1.86